

Package ‘shallot’

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Description Implementations are provided for the models described in the paper D. B. Dahl, R. Day, J. Tsai (2017) <DOI:10.1080/01621459.2016.1165103>. The Ewens, Ewens-Pitman, Ewens attraction, Ewens-Pitman attraction, and ddCRP distributions are available for prior and posterior simulation. Posterior simulation is based on a user-supplied likelihood. Supporting functions for partition estimation and plotting are also provided.

URL <https://github.com/dbdahl/shallot>

BugReports <https://github.com/dbdahl/shallot/issues>

Imports rscala (>= 3.2.18), commonsMath (>= 1.2.5), salso (>= 0.1.16)

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shallot-package*Random Partition Distribution Indexed by Pairwise Information*

Description

This package implements models described in the paper Dahl, D. B., Day, R., and Tsai, J. (2017), **Random Partition Distribution Indexed by Pairwise Information**, *Journal of the American Statistical Association*, 112, 721-732. The Ewens, Ewens-Pitman, Ewens attraction, Ewens-Pitman attraction, and ddCRP distributions are available for prior simulation. We hope in the future to add posterior simulation with a user-supplied likelihood. Supporting functions for partition estimation and plotting are also planned.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

References

Dahl, D. B., Day, R., and Tsai, J. (2017), Random Partition Distribution Indexed by Pairwise Information, *Journal of the American Statistical Association*, 112, 721-732. <DOI:10.1080/01621459.2016.1165103>

See Also

[ewens.pitman.attraction](#), [sample.partitions](#)

Examples

```

data <- iris[,-ncol(iris)]
truth <- as.integer(iris[,ncol(iris)])
distance <- as.dist(as.matrix(dist(scale(data)))+0.001))

decay <- decay.exponential(temperature=9.0, fixed=TRUE), distance)
permutation <- permutation(n.items=nrow(data), fixed = FALSE)
attraction <- attraction(permutation, decay)
mass <- mass(1.0, fixed = TRUE)
discount <- discount(0.2, fixed = TRUE)

```

```
distribution <- ewens.pitman.attraction(mass, discount, attraction)

raw <- sample.partitions(distribution, 500, parallel=FALSE)
samples <- process.samples(raw)

library(salso)
pp <- psm(samples$labels)
est <- salso(pp)
conf <- confidence(est$estimate, pp)
plot(conf)
plot(conf, data=data)
```

association.matrix *Association Matrix*

Description

This function creates an association matrix for a clustering/partition. The (i,j) element of the matrix is 1 if item i and j are in the same cluster/subset and 0 otherwise.

Usage

```
association.matrix(cl)
```

Arguments

cl A vector containing cluster labels for a clustering/partition.

Value

A matrix of 0s and 1s indicating whether items i and j are in the same cluster/subset.

Examples

```
cl <- rep(1:3,times=c(2,4,3))
association.matrix(cl)
```

attraction*Attraction*

Description

This function creates an attraction from a permutation and a decay in preparation for use in the `ewens.attraction`, `ewens.pitman.attraction`, and `ddcrp` functions. For details on each of these arguments, please see the links below.

Usage

```
attraction(permuation, decay)

## S3 method for class 'shallot.attraction'
print(x, ...)

## S3 method for class 'shallot.attraction'
as.matrix(x, ...)
```

Arguments

<code>permuation</code>	An object of class <code>shallot.permutation</code> encoding the permutation of the items.
<code>decay</code>	An object of class <code>shallot.decay</code> detailing the transformation from distances to attractions.
<code>x</code>	An object of class <code>shallot.attraction</code> .
<code>...</code>	Currently ignored.

Value

An object of class `shallot.attraction`.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

References

Dahl, D. B., Day, R., and Tsai, J. (2017), Random Partition Distribution Indexed by Pairwise Information, *Journal of the American Statistical Association*, 112, 721-732. <DOI:10.1080/01621459.2016.1165103>

See Also

`ddcrp`, `decay`, `ewens.attraction`, `ewens.pitman.attraction`, `permuation`

Examples

```
permutation <- permutation(n.items=50, fixed=FALSE)
decay <- decay.exponential(temperature(1.0), dist(scale(USArrests)))
attraction(permutation, decay)
```

decay.reciprocal *Decay Functions*

Description

These functions specify the decay to map distances to attractions.

Usage

```
decay.reciprocal(temperature, distance)

decay.exponential(temperature, distance)

decay.subtraction(temperature, distance, multiplier = 1.01)

## S3 method for class 'shallot.decay'
print(x, ...)
```

Arguments

temperature	An object of class shallot.temperature.
distance	An object of class dist.
multiplier	An scalar greater than 1.0 to ensure that attractions from decay.subtraction are finite.
x	An object of class shallot.decay.
...	Currently ignored.

Details

There are currently three choices for decay functions: reciprocal, exponential, and subtraction.

The reciprocal decay maps a distance d to an attraction a as follows: $a = 1/d^t$, where t is the temperature.

The exponential decay maps a distance d to an attraction a as follows: $a = \exp(-t*d)$, where t is the temperature.

The subtract decay maps a distance d to an attraction a as follows: $a = (m-d)^t$, where t is the temperature and m is the maximum distance in distance multiplied by the supplied *multiplier*.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

References

Dahl, D. B., Day, R., and Tsai, J. (2017), Random Partition Distribution Indexed by Pairwise Information, *Journal of the American Statistical Association*, 112, 721-732. <DOI:10.1080/01621459.2016.1165103>

See Also

[dist](#), [temperature](#), [attraction](#)

Examples

```
temp <- temperature(1.0)
distance <- dist(scale(USArrests))
decay1 <- decay.reciprocal(temp,distance)
decay2 <- decay.exponential(temp,distance)
decay3 <- decay.subtraction(temp,distance)
```

default.mass

Default Mass Selection

Description

This function selects an optimal mass value for Cluster Analysis via Random Partition Distribtuions, using the Ewens-Pitman Attraction distribution.

Usage

```
default.mass(
  mass,
  list.epam,
  dis,
  new.draws = TRUE,
  w = c(1, 1, 1),
  discount = 0,
  temp = 10,
  loss = "binder",
  n.draws = 100L,
  two.stage = TRUE,
  parallel = TRUE
)
## S3 method for class 'shallot.default.mass'
print(x, ...)
```

Arguments

<code>mass</code>	optional, a vector of mass values.
<code>list.epam</code>	optional, a list of expected pairwise allocation matrices. Each matrix in the list needs the attributes "mass" and "n.draws".
<code>dis</code>	a dissimilarity structure of class <code>dist</code> .
<code>new.draws</code>	logical; if TRUE then new draws are obtained at each mass value.
<code>w</code>	a vector of length 3 of the weights to be used in the mass.algorithm .
<code>discount</code>	parameter of the Ewens-Pitman Attraction distribution.
<code>temp</code>	temperature parameter of the Ewens-Pitman Attraction distribution.
<code>loss</code>	One of "binder" or "VI.1b" to indicate the optimization should seek to minimize the expectation of the Binder loss (Binder 1978) or the lower bound of the expectation of the variation of information loss (Wade & Ghahramani 2017), respectively.
<code>n.draws</code>	number of draws of partitions to be obtained at each mass value.
<code>two.stage</code>	logical; if TRUE, the two stage algorithm is implemented in mass.algorithm .
<code>parallel</code>	logical; if TRUE computations will take advantage multiple CPU cores.
<code>x</code>	An object from the default.mass function.
<code>...</code>	currently ignored

Details

The function draws `n.draws` partitions at each specified mass value. If a vector of mass values is not given, then the default of `seq(0.1, 10, 0.2)` is used for loss "VI.1b" and `seq(0.1, 5, 0.05)` used for the other loss functions.

If a list of expected pairwise allocation matrices (EPAM) is provided, additional draws at matching mass values are added to the corresponding matrix. Additionally, no new draws are needed for estimation, if a list of EPAMs is provided.

A partition/clustering estimate from each EPAM is obtained using the SALSO method in [salso](#). The estimate given minimizes the specified loss function with respect to the EPAM.

The function then uses the [mass.algorithm](#) to select the optimal mass value for clustering estimation.

Value

An object of class `shallot.default.mass`. This object is a list containing a matrix of 'best' possible mass values to maximize partition confidence and minimize the variance ratio, the clustering estimate, the expected pairwise allocation matrix, parameters used for optimization and the EPA distribution, and the list of expected pairwise allocation matrices for each mass value.

See Also

Other Default Mass Selection: [mass.algorithm\(\)](#), [partition.confidence\(\)](#), [variance.ratio\(\)](#)

Description

These functions specify the Ewens, Ewens-Pitman, Ewens attraction, Ewens-Pitman attraction, and ddCRP distributions which would then be used in the [sample.partitions](#) function.

Usage

```
ewens(mass, n.items, names = paste0("c", 1:n.items))

## S3 method for class 'shallot.distribution.ewens'
print(x, ...)

ewens.pitman(mass, discount, n.items, names = paste0("c", 1:n.items))

## S3 method for class 'shallot.distribution.ewensPitman'
print(x, ...)

ewens.attraction(mass, attraction)

## S3 method for class 'shallot.distribution.ewensAttraction'
print(x, ...)

ewens.pitman.attraction(mass, discount, attraction)

## S3 method for class 'shallot.distribution.ewensPitmanAttraction'
print(x, ...)

ddcrp(mass, attraction)

## S3 method for class 'shallot.distribution.ddcrp'
print(x, ...)
```

Arguments

mass	An object of class <code>shallot.mass</code> .
n.items	An integer containing the number of items to partition.
names	A character vector containing the names of the items. The default names are of the form “c1”, “c2”, etc.
x	An object of class <code>shallot.distribution</code> .
...	Currently ignored.
discount	An object of class <code>shallot.discount</code> .
attraction	An object of class <code>shallot.attraction</code> .

Value

An object of class `shallot.distribution`.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

References

Dahl, D. B., Day, R., and Tsai, J. (2017), Random Partition Distribution Indexed by Pairwise Information, *Journal of the American Statistical Association*, 112, 721-732. <DOI:10.1080/01621459.2016.1165103>

See Also

`mass`, `discount`, `attraction`, `sample.partitions`

Examples

```
pd1 <- ewens(mass(1),50)

decay <- decay.exponential(temperature(1.0),dist(scale(USArrests)))
attraction <- attraction(permute(n.items=50,fixed=FALSE), decay)
pd2 <- ewens.pitman.attraction(mass(1), discount(0.05), attraction)

pd3 <- ddcrp(mass(1), attraction)
```

mass

Mass, Discount, and Temperature Parameters

Description

These functions set the mass, discount, and temperature parameters and, in the case of them being random, specify the parameters of their distribution.

Usage

```
mass(..., fixed = TRUE)

## S3 method for class 'shallot.mass'
print(x, ...)

discount(..., fixed = TRUE)

## S3 method for class 'shallot.discount'
print(x, ...)
```

```
temperature(..., fixed = TRUE)

## S3 method for class 'shallot.temperature'
print(x, ...)
```

Arguments

...	A number greater than 0.0 representing the value of the mass, discount, or temperature parameters. Or, in the case of them being random, a vector of two numbers representing either: i. the shape and rate parameters of the gamma distribution for the mass or temperature, or ii. the shape parameters of the beta distribution for the discount. This argument is currently ignored for the associated print functions.
fixed	If TRUE, the parameter is fixed. If FALSE, the parameter value is samples from either: i. a gamma distribution for the mass or temperature, or ii. a beta distribution for the discount.
x	An object from the <code>mass</code> , <code>discount</code> , or <code>temperature</code> functions.

Details

If no parameters are specified, the mass parameter defaults to 1.2 , the discount parameter defaults to 0.05 , the temperature parameter defaults to 3.0 . If the mass parameter is random, the default shape and rate parameters of the gamma distribution are 2.5 and 2 , respectively. If the discount parameter is random, the default shape parameters of the beta distribution are 1.0 and 1.0 . If the temperature parameter is random, the default shape and rate parameters of the gamma distribution are 2 and 0.5 , respectively.

Value

An object of class `shallot.mass`, `shallot.discount`, or `shallot.temperature`.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

Examples

```
mass()
mass(1.0)
mass(1.4, fixed=FALSE)
mass(0.5, 1, fixed=FALSE)
discount()
discount(0.2)
discount(1, 3, fixed=FALSE)
temperature()
temperature(2)
temperature(2, 4, fixed=FALSE)
```

<code>mass.algorithm</code>	<i>Mass Selection Algorithm</i>
-----------------------------	---------------------------------

Description

This function selects the optimal mass value for Cluster Analysis via Random Partition distributions using the Ewens-Pitman attraction distribution.

Usage

```
mass.algorithm(mass, pc, vr, n, w = c(1, 1, 1), two.stage = TRUE)
```

Arguments

<code>mass</code>	a vector of mass values
<code>pc</code>	a vector of partition confidences for the partition estimates at the corresponding mass values
<code>vr</code>	a vector of variance ratios for the partition estimates at the corresponding mass values
<code>n</code>	a vector of the number of subsets in the partition estimates at the correpsonding mass values
<code>w</code>	a vector of length 3 specifying the weights of <code>pc</code> , <code>vr</code> , and <code>n</code>
<code>two.stage</code>	logical; if TRUE, the two stage algorithm is implemented

Details

The `mass.algorithm` function is used internally in the `default.mass` function.

The default value for `w` is `c(1, 1, 1)`.

The general algorithm is as follows:

1. Rank the partition confidence (`pc`) and variance ratio (`vr`). Select the `mass_i` value which minimizes the weighed sum of $w_1pc_i + w_2vr_i + w_3n_i$.

The two stage algorithm proceeds as follows:

1. Rank the partition confidence (`pc`) and variance ratio (`vr`). For each number of clusters `n` select the index which minimizes the weighed sum of $w_1pc_i + w_2vr_i$.
2. Rerank the `pc` and `vr` of the selected indices and select the `mass_i` value which minimizes the weighed sum of $w_1pc_i + w_2vr_i + w_3n_i$ from among the selected indices.

Value

A matrix containing the ‘best’ mass value and corresponding values for `pc`, `vr`, and `n`. The matrix also contains the mass values for the partitions estimate with more one more and one less subset than the selected mass value.

See Also

Other Default Mass Selection: [default.mass\(\)](#), [partition.confidence\(\)](#), [variance.ratio\(\)](#)

<code>nsubsets.random</code>	<i>Number of Subsets</i>
------------------------------	--------------------------

Description

These functions either sample the number of subsets for supported partition distributions or computes probabilities, means, and variances of these distributions.

Usage

```
nsubsets.random(x, n.samples)
nsubsets.probability(x, n.subsets)
nsubsets.average(x)
nsubsets.variance(x)
```

Arguments

- `x` An object of class `shallot.distribution`.
- `n.samples` An integer containing the number of samples.
- `n.subsets` An integer containing the number of subsets.

Value

The `nsubsets.random` function returns a vector of random samples of the number of subsets in the distribution `x`.

The `nsubsets.probability` function returns the probability that the number of subsets is `n.subsets` in the distribution `x`. Depending on the number of items and the value of `n.subsets`, this function can be computationally intensive.

The `nsubsets.average` and `nsubsets.variance` functions return the mean and variances, respectively, of the number of subsets in the distribution `x`.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

References

Dahl, D. B., Day, R., and Tsai, J. (2017), Random Partition Distribution Indexed by Pairwise Information, *Journal of the American Statistical Association*, 112, 721-732. <DOI:10.1080/01621459.2016.1165103>

See Also

[partition.distribution](#)

Examples

```
pd <- ewens.pitman.attraction(
  mass(1),
  discount(0.05),
  attraction(permuation(n.items=50,fixed=FALSE),
  decay.exponential(temperature(1.0),dist(scale(USArrests))))))
mean(nsubsets.random(pd,1000))
nsubsets.average(pd)

pde <- ewens(mass(1),50)
nsubsets.variance(pde)
nsubsets.probability(pde,4)
```

partition.confidence Partition Confidence

Description

This function calculates the partition confidence of a partition estimate from the corresponding expected pairwise allocation matrix (EPAM).

Usage

```
partition.confidence(x, y)
```

Arguments

- x If y is not specified then x must be an object of class `salso.confidence`. Otherwise, x is a vector of cluster labels and y is an expected pairwise allocation matrix.
- y If y is not specified then x must be an object of class `salso.confidence`. Otherwise, x is a vector of cluster labels and y is an expected pairwise allocation matrix.

Details

The `partition.confidence` takes as input an object of class `salso.confidence` and then calculates the partition confidence from the expected pairwise allocation matrix.

The partition confidence is the average values of the EPAM for items that are clustered together. Items which are in their own subset do not contribute to partition confidence.

Value

A vector of partition confidences.

See Also

Other Default Mass Selection: [default.mass\(\)](#), [mass.algorithm\(\)](#), [variance.ratio\(\)](#)

Examples

```
x <- rep(c(1,2,3), times=c(2,3,5))
y <- diag(10)
y[upper.tri(y)] <- runif(45)
partition.confidence(x,y)
```

partition.pmf

Obtain the Probability Mass Function of a Partition Distribution

Description

This function returns the probability mass function (pmf) of a partition distribution.

Usage

```
partition.pmf(x)
```

Arguments

x	An object of class <code>shallot.distribution</code> obtained, for example, from the ewens.pitman.attraction function.
---	----------------------------------------------------------------------------------------------------------------------------------------

Value

A function that takes a partition (as a vector in cluster label notation) and returns the probability — or, if `log=TRUE`, the log of the probability — of the supplied partition.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

Examples

```
## Not run:
example(shallot)

## End(Not run)
```

permutation	<i>Permutation</i>
-------------	--------------------

Description

These function define a permutation for subsequent use.

Usage

```
permutation(..., n.items = NULL, fixed = TRUE)

## S3 method for class 'shallot.permutation'
print(x, ...)
```

Arguments

...	For the function <code>permutation</code> , a permutation of the integers 1, 2,... n, where n is the length of the vector. For the function <code>print.shallot.permutation</code> , this is ignored.
n.items	An optional argument provided instead of ... to request a random partition. The argument <code>fixed</code> must be FALSE.
fixed	Should the permutation be fixed?
x	An object of class <code>shallot.permutation</code> .

Details

A valid permutation of length n is an integer vector of length n containing each integer 1, 2,... n only once.

Value

An object of class `shallot.permutation`.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

References

Dahl, D. B., Day, R., and Tsai, J. (2017), Random Partition Distribution Indexed by Pairwise Information, *Journal of the American Statistical Association*, 112, 721-732. <DOI:10.1080/01621459.2016.1165103>

See Also

`attraction`

Examples

```
## Demonstrate permutation.
permutation(c(3, 1, 2, 5, 4))
permutation(c(3, 1, 2, 5, 4), fixed=FALSE)
permutation(n.items=5, fixed=FALSE)
```

process.samples *Process Sampled Partitions*

Description

This function extracts the partitions from the results of the [sample.partitions](#) function.

Usage

```
process.samples(x)
```

Arguments

x An object from the [sample.partitions](#) function.

Details

This function extracts the sampled partitions from the results of the [sample.partitions](#) function.

Value

A list containing a matrix of cluster labels in which each row represents a clusterings. The list also contains sampled model parameters if [sample.parameter](#) is not NULL.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

See Also

[sample.partitions](#)

Examples

```
## Not run:
example(shallot)

## End(Not run)
```

sample.partitions	<i>Sample Partitions from Partition Distributions</i>
-------------------	-------------------------------------------------------

Description

This function samples partitions from the Ewens, Ewens-Pitman, Ewens attraction, Ewens-Pitman attraction, and ddCRP distributions.

Usage

```
sample.partitions(x, n.draws, parallel = TRUE)
```

Arguments

- | | |
|----------|-------------------------------------------------------------------------------------------------------------------------------------|
| x | An object of class <code>shallot.distribution</code> obtained, for example, from the <code>ewens.pitman.attraction</code> function. |
| n.draws | An integer representing the desired number of samples. Due to parallelization, slightly more samples may be returned. |
| parallel | Should sampling be done in parallel by simultaneously using all CPU cores? |

Value

An object of class `shallot.samples.raw` which can be subsequently be used in `process.samples`.

Note

If this function is interrupted by the user, the computation engine will be broken and subsequent calls to package functions may fail until a new session is started.

Author(s)

David B. Dahl <dahl@stat.byu.edu>

See Also

`partition.distribution`, `process.samples`

Examples

```
## Not run:  
example(shallot)  
  
## End(Not run)
```

variance.ratio *Variance Ratio*

Description

This function calculates the variance of the expected pairwise allocation matrix (EPAM) within clusters/subsets over the total variance of the expected pairwise allocation matrix.

Usage

```
variance.ratio(x, y)
```

Arguments

<code>x, y</code>	If <code>y</code> is not specified then <code>x</code> must be an object of class <code>salso.confidence</code> . Otherwise, <code>x</code> is a vector of cluster labels and <code>y</code> is an expected pairwise allocation matrix.
-------------------	-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

Details

The `variance.ratio` function takes as input an object of class `salso.confidence` and calculates the variance ratio for the estimated partition from the corresponding expected pairwise allocation matrix (EPAM).

The variance ratio is the weighted average of the within cluster variances of the EPAM, weighted by the number of pairwise EPAM values per cluster, over the total variance of the EPAM.

Value

A vector of variance ratios.

See Also

Other Default Mass Selection: `default.mass()`, `mass.algorithm()`, `partition.confidence()`

Examples

```
x <- rep(c(1,2,3), times=c(2,3,5))
y <- diag(10)
y[upper.tri(y)] <- runif(45)
variance.ratio(x,y)
```

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