A note on the rcc function in the rQCC package

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Abstract

In this note, we provide a brief summary of control charts and a description of how they are constructed using the rcc function in the robust quality control chart (rQCC) R package. Using rcc function, one can construct the traditional Shewhart-type control charts. In addition, using various robust location and scale estimates provided by the rQCC package, one can easily obtain robust alternatives to the traditional charts.

1 Introduction

Control charts, also known as Shewhart control charts [1, 2, 3], have been widely used to monitor whether a manufacturing process is in a proper state of control or not. The traditional Shewhart-type control charts are made up of the upper control limit (UCL), the center line (CL) and the lower control limit (LCL) and they have the form of $\text{CL} \pm g \cdot \text{SE}$, where the American Standard is based on g=3 with a target false alarm rate of 0.027% and the British Standard is based on g=3.09 with a target false alarm rate of 0.020%. The UCL is given by $\text{CL} + g \cdot \text{SE}$ and the LCL is $\text{CL} - g \cdot \text{SE}$.

In what follows, we provide how to construct the traditional Shewhart-type control charts and robust alternatives to them using various robust location and scale estimates provided by the rQCC package. In this note, we assume that we have m samples and that each sample has the same sample size of n. Let X_{ij} be the ith sample (subgroup) from a stable manufacturing process, where i = 1, 2, ..., m and j = 1, 2, ..., n. We also assume that X_{ij} are independent and identically distributed as normal with mean μ and variance σ^2 . The A, B and D notations here follow the definitions in ASTM (STP 15-C) [4] and ASTM (STP 15-D) [5].

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2 The \bar{X} chart

In order to construct the $CL \pm g \cdot SE$ control limits, we consider the relation

$$\frac{\bar{X}_k - E(\bar{X}_k)}{\operatorname{SE}(\bar{X}_k)} = \pm g.$$

Since $E(\bar{X}_k) = \mu$ and $SE(\bar{X}_k) = \sigma/\sqrt{n}$, we have

$$E(\bar{X}_k) \pm g \cdot SE(\bar{X}_k) = \mu \pm \frac{g}{\sqrt{n}} \sigma.$$

Then the control limits for the \bar{X} chart are given by

$$\begin{aligned} \text{UCL} &= \mu + A(n)\sigma, \\ \text{CL} &= \mu, \\ \text{LCL} &= \mu - A(n)\sigma, \end{aligned}$$

where $A(n) = g/\sqrt{n}$. In practice, the values of the parameters, μ and σ , are not known. Thus, with the estimates $\hat{\mu}$ and $\hat{\sigma}$, we have

$$\begin{aligned} \text{UCL} &= \hat{\mu} + \frac{g}{\sqrt{n}} \hat{\sigma}, \\ \text{CL} &= \hat{\mu}, \\ \text{LCL} &= \hat{\mu} - \frac{g}{\sqrt{n}} \hat{\sigma}. \end{aligned} \tag{1}$$

Thus, we need to estimate μ and σ by using each sample and then pooling these estimates. Using the *i*th sample above, the sample mean and variance are given by

$$\bar{X}_i = \frac{1}{n} \sum_{i=1}^n X_{ij}$$
 and $S_i^2 = \frac{1}{n-1} \sum_{i=1}^n (X_{ij} - \bar{X}_i)^2$,

where i = 1, 2, ..., m. Then we can estimate μ using all the samples as below:

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_m}{m} = \frac{1}{m} \sum_{i=1}^m \bar{X}_i.$$

Note that it is easily seen that \bar{X} is unbiased for μ . However, S_i is not unbiased for σ since $E(S_i) = c_4(n)\sigma$, where $c_4(n) = \sqrt{2/(n-1)} \cdot \Gamma(n/2)/\Gamma(n/2-1/2)$. Thus, $S_i/c_4(n)$ is unbiased for σ . Then we can easily show that $\bar{S}/c_4(n)$ is unbiased for σ , where

$$\bar{S} = \frac{1}{m} \sum_{i=1}^{m} S_i.$$

Thus, by substituting $\hat{\mu} = \bar{\bar{X}}$ and $\hat{\sigma} = \bar{S}/c_4(n)$ into (1), we have the control limits

$$UCL = \bar{X} + \frac{g}{\sqrt{n}} \frac{\bar{S}}{c_4(n)} = \bar{X} + A_3(n)\bar{S},$$

$$CL = \bar{X},$$

$$LCL = \bar{X} - \frac{g}{\sqrt{n}} \frac{\bar{S}}{c_4(n)} = \bar{X} - A_3(n)\bar{S},$$

where $A_3(n) = A(n)/c_4(n) = g/\{c_4(n)\sqrt{n}\}.$

It is also known that

$$E(R) = d_2(n)\sigma,$$

where R is the sample range from $X_i \sim N(\mu, \sigma^2)$ and

$$d_2(n) = 2 \int_0^\infty \left\{ 1 - \left[\Phi(z) \right]^n - \left[1 - \Phi(z) \right]^n \right\} dz.$$

For more details on $d_2(n)$, one can refer to the vignette below.

Then, with the ith sample, $R_i/d_2(n)$ is unbiased for σ , where

$$R_i = \max_{1 \le j \le n} (X_{ij}) - \min_{1 \le j \le n} (X_{ij}).$$

Then, with the m samples, $\bar{R}/d_2(n)$ is unbiased for σ , where

$$\bar{R} = \frac{1}{m} \sum_{i=1}^{m} R_i.$$

Substituting $\hat{\mu} = \bar{X}$ and $\hat{\sigma} = \bar{R}/d_2(n)$ into (1), we have the control limits

$$UCL = \bar{X} + \frac{g}{\sqrt{n}} \frac{\bar{R}}{d_2(n)} = \bar{X} + A_2(n)\bar{R},$$

$$CL = \bar{X},$$

$$LCL = \bar{X} - \frac{g}{\sqrt{n}} \frac{\bar{R}}{d_2(n)} = \bar{X} - A_2(n)\bar{R},$$

where $A_2(n) = A(n)/d_2(n) = g/\{d_2(n)\sqrt{n}\}.$

As alternatives to the above, we can use robust estimates of location and scale. For example, using the median, we can estimate μ

$$\hat{\mu} = \frac{M_1 + M_2 + \dots + M_m}{m} = \frac{1}{m} \sum_{i=1}^{m} M_i,$$

where

$$M_i = \underset{1 \le j \le n}{\operatorname{median}}(X_{ij}).$$

One can also consider estimating σ based on the conventional MAD (median absolute deviation) given by

$$\mathrm{MAD} = \frac{\underset{1 \leq i \leq n}{\mathrm{median}} |X_i - M|}{\Phi^{-1}(3/4)} \approx 1.4826 \cdot \underset{1 \leq i \leq n}{\mathrm{median}} |X_i - M|,$$

where $X_i \sim N(\mu, \sigma^2)$ and $M = \text{median}(X_i)$. Here $\Phi^{-1}(3/4)$ is needed to make this estimator Fisher-consistent [6] for the standard deviation under the normal distribution. For more details, see the references [7, 8]. It should be noted that the above conventional MAD estimator is Fisher-consistent but not unbiased. The "unbiased MAD" (uMAD) with a finite sample is developed by Park, Kim and Wang [8] and implemented in the rQCC package (see mad.unbiased function).

Then, with the m samples, we have the robust unbiased estimate of σ as follows

$$\hat{\sigma} = \frac{\text{uMAD}_1 + \text{uMAD}_2 + \dots + \text{uMAD}_m}{m} = \frac{1}{m} \sum_{i=1}^{m} \text{uMAD}_i,$$

where

$$uMAD_i = uMAD_{i \le j \le n}(X_{ij}).$$

 $\mathrm{uMAD}_i = \mathrm{uMAD}_i(X_{ij}).$ The rcc function constructs the control charts based on various unbiased estimates. For example, with the median and uMAD estimates, one can obtain the control limits using the following

Another way of constructing the control limits is to use the Hodges-Lehmann [9] for location and Shamos [10] for scale which are respectively given by

$$HL = \operatorname{median}\left(\frac{X_i + X_j}{2}\right)$$

and

$$\operatorname{Shamos} = \frac{\underset{i < j}{\operatorname{median}} (|X_i - X_j|)}{\sqrt{2} \Phi^{-1}(3/4)} \approx 1.048358 \cdot \underset{i < j}{\operatorname{median}} (|X_i - X_j|),$$

where $\sqrt{2}\Phi^{-1}(3/4)$ is needed to make Shamos estimator Fisher-consistent for the standard deviation under the normal distribution [11]. For the Hodges-Lehmann estimate, the median is obtained by three ways: (i) the pairwise averages with i < j (denoted by HL1), (ii) the pairwise averages with $i \leq j$ (HL2), and (iii) all the pairwise averages (HL3). For more details, refer to [8]. It should be noted that the above Shamos is Fisher-consistent but not unbiased. The Hodges-Lehmann and "unbiased Shamos" are also developed by [8] and implemented in R (see HL and shamos.unbiased). For example, with the HL2 and unbiased Shamos estimates, one can obtain the control limits as below.

As shown above, by choosing the options for loc and scale, one can construct various control charts.

3 The S chart

In order to construct the CL $\pm g \cdot$ SE control limits, we can consider the relation

$$\frac{S_k - E(S_k)}{\operatorname{SE}(S_k)} = \pm g.$$

Since $E(S_k) = c_4(n)\sigma$ and $SE(S_k) = \sqrt{1 - c_4(n)^2} \cdot \sigma$, we have

$$E(S_k) \pm g \cdot SE(S_k) = \left\{ c_4(n) \pm g\sqrt{1 - c_4(n)^2} \right\} \sigma.$$

The control limits for the S chart are given by

UCL =
$$B_6(n)\sigma$$
,
CL = $c_4(n)\sigma$,
LCL = $B_5(n)\sigma$,

where

$$B_5(n) = \max \left\{ c_4(n) - g \cdot \sqrt{1 - c_4(n)^2}, \ 0 \right\},$$

$$B_6(n) = c_4(n) + g \cdot \sqrt{1 - c_4(n)^2}.$$

Since σ is unknown in practice, we need to choose an appropriate unbiased estimate for σ . One can consider $\hat{\sigma} = \bar{S}/c_4(n)$. Then we have

$$UCL = B_4(n)\bar{S},$$

$$CL = \bar{S},$$

$$LCL = B_3(n)\bar{S},$$

where $B_3(n) = B_5(n)/c_4(n)$ and $B_4(n) = B_6(n)/c_4(n)$.

To obtain the robustness property, one can consider a robust estimate of σ . For example, the unbiased MAD or unbiased Shamos estimates of σ can be used as seen before. The limits for the S chart are calculated using the rcc function with type="S" as below.

4 The R chart

We consider the relation

$$\frac{R_k - E(R_k)}{\operatorname{SE}(R_k)} = \pm g.$$

Since $E(R_k) = d_2(n)\sigma$ and $Var(R_k) = d_3(n)^2\sigma^2$, we have

$$E(R_k) \pm g \cdot SE(R_k) = \{d_2(n) \pm gd_3(n)\}\sigma.$$

The control limits for the R chart are given by

UCL =
$$D_2(n)\sigma$$
,
CL = $d_2(n)\sigma$,
LCL = $D_1(n)\sigma$,

where

$$D_1(n) = \max \{d_2(n) - g \cdot d_3(n), 0\},$$

$$D_2(n) = d_2(n) + g \cdot d_3(n),$$

Since σ is unknown in practice, we need to choose an appropriate unbiased estimate for σ . One can consider $\hat{\sigma} = \bar{R}/d_2(n)$. Then we have

$$UCL = D_4(n)\bar{R},$$

$$CL = \bar{R},$$

$$LCL = D_3(n)\bar{R},$$

where $D_3(n) = D_1(n)/d_2(n)$ and $D_4(n) = D_2(n)/d_2(n)$. These limits are easily calculated using the rcc function as below.

As afore-mentioned, we can consider a robust estimate of σ . For example, the control limits with the unbiased Shamos are calculated as below.

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