A Quick Guide for the QZ Package

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pbdR Core Team

Contents

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Warning: This document is written to explain the main functions of **QZ** package (Chen 2013), version 0.1-3. Every effort will be made to ensure future versions are consistent with these instructions, but features in later versions may not be explained in this document.

1. Introduction

This article is to explain the **QZ** package (Chen 2013), and is organized as the following. Section 2 introduces briefly background of generalized eigenvalues problem and QZ decomposition. Section 3 lists the main functions and detail Fortran functions of LAPACK library (Anderson *et al.* 1999).

2. Methods

Some details can be found on wikipedia website at

`http://en.wikipedia.org/wiki/Eigendecomposition_of_a_matrix`

for generalized eigenvalues, and at

`http://en.wikipedia.org/wiki/Schur_decomposition`

about QZ decomposition or generalized Schur form. The LAPACK (Anderson *et al.* 1999) also provides functions to solve these problems.

2.1. Generalized Eigenvalues for Pair Matrices

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $N \times N$ non-symmetric matrices which can be both in real or in complex. The goal is to find right generalized eigen vectors $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{B}\boldsymbol{v}$, or left generalized eigen vectors $\boldsymbol{u}$ such that $\boldsymbol{u}^H\boldsymbol{A} = \lambda\boldsymbol{u}^H\boldsymbol{B}$ where $\boldsymbol{u}^H$ is the conjugate-transpose of $\boldsymbol{u}$. Also, λ is called generalized eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ which obeys $\det(\boldsymbol{A} - \lambda\boldsymbol{B}) = 0$. Note that λ, $\boldsymbol{u}$, and $\boldsymbol{v}$ may be complex even $\boldsymbol{A}$ and $\boldsymbol{B}$ are in real.

Suppose $\boldsymbol{B}$ is an identity matrix $\boldsymbol{I}$, then the problem reduces to traditional eigenvalue problem. i.e. This is a special case.

2.2. QZ Decomposition for Pair Matrices

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $N \times N$ non-symmetric matrices which can be both in real or in complex. The QZ decomposition factorizes both matrices as

- $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Z}^\top$ and $\boldsymbol{B} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Z}^\top$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are real, or
- $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Z}^H$ and $\boldsymbol{B} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Z}^H$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are complex

where $\boldsymbol{Q}$ and $\boldsymbol{Z}$ are unitary and $\boldsymbol{S}$ and $\boldsymbol{T}$ are upper triangular. The unitary means $\boldsymbol{X}\boldsymbol{X}^H = \boldsymbol{I}$ if $\boldsymbol{X}$ is complex or $\boldsymbol{X}\boldsymbol{X}^\top = \boldsymbol{I}$ if $\boldsymbol{X}$ is real where $\boldsymbol{I}$ is the identity matrix.

The QZ decomposition is also called generalized Schur decomposition where $\boldsymbol{S}$ and $\boldsymbol{T}$ are the Schur form of $\boldsymbol{A}$ and $\boldsymbol{B}$. The generalized eigenvalues λ that solve the generalized eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{B}\boldsymbol{x}$ where $\boldsymbol{x}$ is an unknown nonzero vector and $\lambda_i = \boldsymbol{S}_{ii}/\boldsymbol{T}_{ii}$.

Suppose $\boldsymbol{B}$ is an identity matrix $\boldsymbol{I}$, then the problem reduces to fine $\boldsymbol{Q}$ such that $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Q}^{-1}$ for real $\boldsymbol{A}$ or $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Q}^{H}$ for complex $\boldsymbol{A}$. i.e. This is a special case.

3. Implementation

Two main functions are `geigen()` for generalized eigenvalues, and `qz()` for QZ decomposition with reordering capability. Both functions are able to deal a single matrix $\boldsymbol{A}$ or a paired matrices $(\boldsymbol{A}, \boldsymbol{B})$ in both complex and real systems. Both functions are wrapper functions for several lower level R functions `qz.*()` which are also wrapper functions via `.Call()` for C and Fortran functions to LAPACK library version 3.4.2.

3.1. R and LAPACK Functions

In the **QZ** package, `qz()` will based on the data type of the input matrix $\boldsymbol{A}$ or the paired matrices $(\boldsymbol{A}, \boldsymbol{B})$ to select accordingly the warpper functions `qz.*()` as the feature of S3 method in R.

In R, one may still used the qz.*() individually if needed, or convert the data type as desired. For example, one may use `as.complex()` on the input real/double matrix $\boldsymbol{A}$ or matrices $(\boldsymbol{A}, \boldsymbol{B})$ to call the complex version of `qz()` as below

Use complex system via `as.complex()`

```
set.seed(123)
A <- matrix(rnorm(9), nrow = 3)
B <- matrix(rnorm(9), nrow = 3)
ret.d <- qz(A, B, only.values = TRUE)
ret.c <- qz(matrix(as.complex(A), nrow(A)),
            matrix(as.complex(B), nrow(B)), only.values = TRUE)
str(ret.d)  # class of "dgges"
str(ret.c)  # class of "zgges"
```

LAPACK library is internally incorporated in **QZ** including complex*16 and double precision for complex and real systems, respectively. To use internal LAPACK library of the **QZ**, one may add `-configure-args="-enable-iqz"` to `R CMD INSTALL QZ_*.tar.gz` when installing the **QZ** package. **QZ** also allows some functions of LAPACK and BLAS (Blackford *et al.* 2002) independently to the R's LAPACK and BLAS libraries when some functions are not available. Table 1 provides a detail lists for the `qz.*()` functions.

3.2. Reording

An extral MATLAB-like function `ordqz()` is also available to reordering generalized eigenvalues and QZ decomposition results. The function which is the combinations of `qz()` and `qz.ztrsen()/qz.dtrsen()` for specified ordering keywords in Table 2. The keywords `lhp`, `rhp`, `udi`, and `udo` are implemented as (or similar to) the way Matlab does. Additionally, the keywords `*.fo` are implemented to select finite (generalized) eigen values only. Note that `select` argument of `qz()` allows users to specify any order to group and reorder the decompositions.

Table 1: **QZ** functions

Function	Wrapper	Main Input	System	Purpose
`geigen()`	`qz.zgeev`	$\boldsymbol{A}$	Complex	Generalized eigenvalues
	`qz.dgeev`	$\boldsymbol{A}$	Real	
`qz()`	`qz.zgees`	$\boldsymbol{A}$	Complex	QZ decomposition
	`qz.dgees`	$\boldsymbol{A}$	Real	
	`qz.ztrsen`	$\boldsymbol{T}, \boldsymbol{Q}$	Complex	Reordering
	`qz.dtrsen`	$\boldsymbol{T}, \boldsymbol{Q}$	Real	
`geigen()`	`qz.zggev`	$(\boldsymbol{A}, \boldsymbol{B})$	Complex	Generalized eigenvalues
	`qz.dggev`	$(\boldsymbol{A}, \boldsymbol{B})$	Real	
`qz()`	`qz.zgges`	$(\boldsymbol{A}, \boldsymbol{B})$	Complex	QZ decomposition
	`qz.dgges`	$(\boldsymbol{A}, \boldsymbol{B})$	Real	
	`qz.ztgsen`	$(\boldsymbol{S}, \boldsymbol{T}), \boldsymbol{Q}, \boldsymbol{Z}$	Complex	Reordering
	`qz.dtgsen`	$(\boldsymbol{S}, \boldsymbol{T}), \boldsymbol{Q}, \boldsymbol{Z}$	Real	

Table 2: The `ordez()` keyword for reording.

`keyword`	Purpose
`lhp`	Left-half (real(E) < 0)
`rhp`	Right-half (real(E) >= 0)
`udi`	Interior of unit disk (abs(E) < 1)
`udo`	Exterior of unit disk (abs(E) >= 1)
`ref`	Real eigenvalues first (top-left conner)
`cef`	Complex eigenvalues first (top-left conner)
`lhp.fo`	Left-half (real(E) < 0) and finite only
`rhp.fo`	Right-half (real(E) >= 0) and finite only
`udi.fo`	Interior of unit disk (abs(E) < 1) and finite only
`udo.fo`	Exterior of unit disk (abs(E) >= 1) and finite only

3.3. For Matlab Users

In Matlab, one may need to specify options for complex or real systems be used for obtaining the (generalized) eigenvalues and for constructing the QZ decomposition. Some Matlab versions may assume a "complex" system as the default regardless input types.

Some Matlab versions may need additional options to turn on "double precision" for the QZ decomposition. The error of the QZ decomposition (in terms of maximum modulus of whole matrix) may be around in the range of `1e-05` to `1e-06` where the modulus is $r = Mod(z) = sqrt(x^2 + y^2)$. Note that `1e-15` to `1e-16` should be expected if double precission is used correctly.

The notations used in Matlab also may not be followed or consistent as used in

- **QZ** in R,
- its depending LAPACK functions,

- general QZ notations in some textbooks, or
- Wikipedia webpage.

In newer Matlab, the documents say that `[AA, BB, Q, Z] = qz(A, B)` produces upper quasitriangular matrices `AA` and `BB`, and unitary matrices `Q` and `Z` such that `Q * A * Z = AA` and `Q * B * Z = BB`. Here, the `Z` in the Matlab notations are the conjugated versions of the QZ decomposition where $A = Q \times S \times Z^*$ and $B = Q \times T \times Z^*$. i.e. The `Z` in Matlab is actually Z^* in the general QZ notations. The `AA` and `BB` in Matlab are the same S and T in the general QZ notations.

4. Data Example

There are four demos for the **QZ** package which are listed in Table 3

Table 3: The demos of **QZ** package.

demo	Purpose
ex1_geigen	geigen() for double/complex single/paired matrices
ex2_qz	qz() for double/complex single/paired matrices
ex3_ordqz	ordqz() and arbiturary reordering
ex4_fda_geigen	generalized eigen analysis of **fda** pacakge (Ramsay *et al.* 2013)

There are also several datasets for **QZ** package to verify results which are listed in Table 4.

Table 4: The datasets of **QZ** package.

data	Source
exAB1	http://www.nag.com/lapack-ex/node124.html
exAB2	http://www.nag.com/lapack-ex/node119.html
exAB3	http://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/F08/f08yuf.xml
exAB4	http://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/F08/f08yuf.xml
exA1	http://www.nag.com/lapack-ex/node94.html
exA2	http://www.nag.com/lapack-ex/node89.html
exA3	http://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/F08/f08quf.xml
exA4	http://www.nag.com/numeric/fl/nagdoc_fl22/xhtml/F08/f08quf.xml

These demos can be obtained in R by the following.

QZ demo ex1_geigen

```
> demo(ex1_geigen, 'QZ')


        demo(ex1_geigen)
        ---- ~~~~~~~~~~
```

```
Type  <Return>   to start :

> library(QZ, quiet = TRUE)

> ### http://www.nag.com/lapack-ex/node122.html
> (ret <- geigen(exAB1$A, exAB1$B))
ALPHA:
[1] 19.03-57.10i 11.88-29.70i 10.96- 3.65i 21.87-27.34i

BETA:
[1] 6.344+0i 5.941+0i 3.654+0i 5.468+0i

U:
                [,1]             [,2]             [,3]               [,4]
[1,]  0.0358-0.1155i  0.0725-0.3001i  0.1650-0.0068i  0.01727-0.02542i
[2,]  0.2152+0.2357i -0.2139+0.7641i  0.0999-0.8330i -0.01045-0.09180i
[3,] -0.2425+0.4271i  0.7520-0.2317i -0.9374-0.0626i -0.17518-0.82482i
[4,]  0.5658-0.4342i -0.1782-0.8218i -0.0529+0.1385i -0.84361-0.01589i

V:
                [,1]                [,2]            [,3]
                   [,4]
[1,] -0.8238-0.1762i  0.63974+0.360259i 0.9775+0.0225i
   -0.90623+0.093766i
[2,] -0.1530+0.0707i  0.00416-0.000547i 0.1591-0.1137i
   -0.00743+0.006875i
[3,] -0.0707-0.1530i  0.04021+0.022645i 0.1209-0.1537i
   0.03021-0.003126i
[4,]  0.1530-0.0707i -0.02264+0.040212i 0.1537+0.1209i
   -0.01459-0.140970i

> ### http://www.nag.com/lapack-ex/node117.html
> (ret <- geigen(exAB2$A, exAB2$B))
ALPHA:
[1] 3.801+0.000i 3.030+4.040i 1.563-2.084i 4.000+0.000i

BETA:
[1] 1.900 1.010 0.521 1.000

U:
            [,1]            [,2]            [,3]           [,4]
[1,]  0.53333+0i  0.2171-0.1284i  0.2171+0.1284i -7.276e-17+0i
[2,] -0.06667+0i  0.1744-0.1851i  0.1744+0.1851i -1.000e+00+0i
[3,] -1.00000+0i -0.7928+0.2072i -0.7928-0.2072i  1.000e+00+0i
[4,]  0.60000+0i  0.3912+0.0911i  0.3912-0.0911i -3.695e-16+0i

V:
            [,1]            [,2]            [,3]         [,4]
[1,] 1.000000+0i -0.4398-0.5602i -0.4398+0.5602i -1.00000+0i
[2,] 0.005714+0i -0.0880-0.1120i -0.0880+0.1120i -0.01111+0i
[3,] 0.062857+0i -0.1424+0.0031i -0.1424-0.0031i  0.03333+0i
[4,] 0.062857+0i -0.1424+0.0031i -0.1424-0.0031i -0.15556+0i
```

```
> ### http://www.nag.com/lapack-ex/node92.html
> (ret <- geigen(exA1$A))
W:
[1] -6.000-7.000i -5.000+2.006i  7.998-0.996i  3.002-4.000i

U:
                [,1]             [,2]             [,3]            [,4]
[1,]  0.8357+0.0000i -0.3510+0.1013i -0.1689+0.2595i 0.1099-0.2007i
[2,] -0.0794+0.3372i -0.4035+0.4540i  0.6762+0.0000i 0.0336+0.2312i
[3,]  0.0917+0.3097i  0.6239+0.0000i  0.3032+0.5642i 0.0944-0.3947i
[4,]  0.0456-0.2741i -0.0816-0.3190i  0.1328+0.1376i 0.8534+0.0000i

V:
                [,1]             [,2]             [,3]             [,4]
[1,]  0.8457+0.0000i -0.3865+0.1732i -0.1730+0.2669i -0.0356-0.1782i
[2,] -0.0177+0.3036i -0.3539+0.4529i  0.6924+0.0000i  0.1264+0.2666i
[3,]  0.0875+0.3115i  0.6124+0.0000i  0.3324+0.4960i  0.0129-0.2966i
[4,] -0.0561-0.2906i -0.0859-0.3284i  0.2504-0.0147i  0.8898+0.0000i

> ### http://www.nag.com/lapack-ex/node87.html
> (ret <- geigen(exA2$A))
W:
[1]  0.7995+0.0000i -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i

U:
            [,1]             [,2]             [,3]        [,4]
[1,] -0.62447+0i  0.5330+0.0000i  0.5330+0.0000i  0.6641+0i
[2,] -0.59949+0i -0.2666+0.4041i -0.2666-0.4041i -0.1068+0i
[3,]  0.49992+0i  0.3455+0.3153i  0.3455-0.3153i  0.7293+0i
[4,] -0.02709+0i -0.2541-0.4451i -0.2541+0.4451i  0.1249+0i

V:
            [,1]             [,2]             [,3]       [,4]
[1,] -0.65509+0i -0.1933+0.2546i -0.1933-0.2546i 0.1253+0i
[2,] -0.52363+0i  0.2519-0.5224i  0.2519+0.5224i 0.3320+0i
[3,]  0.53622+0i  0.0972-0.3084i  0.0972+0.3084i 0.5938+0i
[4,] -0.09561+0i  0.6760+0.0000i  0.6760+0.0000i 0.7221+0i
```

QZ demo ex2_qz

```
> demo(ex2_qz, 'QZ')


        demo(ex2_qz)
        ---- ~~~~~~

Type  <Return>   to start :

> library(QZ, quiet = TRUE)

> ### http://www.nag.com/lapack-ex/node124.html
> (ret <- qz(exAB1$A, exAB1$B))
ALPHA:
```

```
[1] 19.03-57.10i 11.88-29.70i 10.96- 3.65i 21.87-27.34i

BETA:
[1] 6.344+0i 5.941+0i 3.654+0i 5.468+0i
S:
            [,1]         [,2]          [,3]          [,4]
[1,] 19.03-57.1i 53.59-89.82i -81.31-63.23i 106.66-44.79i
[2,]  0.00+ 0.0i 11.88-29.70i   3.56+27.63i  -0.67-16.42i
[3,]  0.00+ 0.0i  0.00+ 0.00i  10.96- 3.65i -25.02- 8.20i
[4,]  0.00+ 0.0i  0.00+ 0.00i   0.00+ 0.00i  21.87-27.34i

T:
         [,1]         [,2]          [,3]          [,4]
[1,] 6.344+0i 3.399+0.712i -0.515-2.382i  6.582+2.430i
[2,] 0.000+0i 5.941+0.000i -2.448-0.343i  5.739-0.702i
[3,] 0.000+0i 0.000+0.000i  3.654+0.000i -1.410-3.933i
[4,] 0.000+0i 0.000+0.000i  0.000+0.000i  5.468+0.000i

Q:
                [,1]             [,2]             [,3]               [,4]
[1,] -0.3347+0.7387i  0.2872-0.4789i  0.1725+0.0093i  0.01443-0.02124i
[2,] -0.1277+0.2493i -0.0282+0.4999i  0.1541-0.8008i -0.00873-0.07670i
[3,] -0.3557+0.0396i -0.4615-0.0822i -0.3939+0.0258i -0.14637-0.68917i
[4,] -0.0126-0.3682i  0.1508-0.4417i  0.1517-0.3555i -0.70486-0.01328i

Z:
                [,1]             [,2]               [,3]
                                      [,4]
[1,] -0.9240-0.1977i  0.2460+0.2090i -0.00543+0.05421i
   0.000e+00+0.000e+00i
[2,] -0.1716+0.0793i -0.5943+0.0905i  0.74673-0.21271i
   -1.092e-16-3.690e-16i
[3,] -0.0793-0.1716i  0.0943-0.5082i  0.01020-0.44383i
   7.034e-01-7.277e-02i
[4,]  0.1716-0.0793i  0.5082+0.0943i  0.44383+0.01020i
   -7.277e-02-7.034e-01i

> ### http://www.nag.com/lapack-ex/node119.html
> (ret <- qz(exAB2$A, exAB2$B))
ALPHA:
[1] 3.801+0.000i 3.030+4.040i 1.563-2.084i 4.000+0.000i

BETA:
[1] 1.900 1.010 0.521 1.000
S:
      [,1]   [,2]    [,3]    [,4]
[1,] 3.801 31.326 -61.485 -66.836
[2,] 0.000  3.351   7.074   6.692
[3,] 0.000 -1.192   1.410   4.379
[4,] 0.000  0.000   0.000   4.000

T:
     [,1]   [,2]    [,3]   [,4]
```

```
[1,]  1.9 -1.078 -5.6252 -9.987
[2,]  0.0  1.176  0.0000  1.751
[3,]  0.0  0.000  0.4474  1.090
[4,]  0.0  0.000  0.0000  1.000

Q:
       [,1]     [,2]    [,3]       [,4]
[1,] 0.4642  0.81159  0.3547 -5.145e-17
[2,] 0.5002 -0.06975 -0.4950 -7.071e-01
[3,] 0.5002 -0.06975 -0.4950  7.071e-01
[4,] 0.5331 -0.57585  0.6198 -2.613e-16

Z:
         [,1]     [,2]     [,3]       [,4]
[1,] 0.996056  0.08183 -0.03428  0.000e+00
[2,] 0.005692 -0.44454 -0.89574  5.145e-17
[3,] 0.062609 -0.63075  0.31343  7.071e-01
[4,] 0.062609 -0.63075  0.31343 -7.071e-01

> ### http://www.nag.com/lapack-ex/node94.html
> (ret <- qz(exA1$A))
W:
[1] -6.000-7.000i -5.000+2.006i  7.998-0.996i  3.002-4.000i

T:
      [,1]             [,2]             [,3]              [,4]
[1,] -6-7i  0.1618+0.4896i 0.4761-0.1946i  0.8633-0.3014i
[2,]  0+0i -5.0000+2.0060i 0.6907+0.2115i  0.2281+0.1328i
[3,]  0+0i  0.0000+0.0000i 7.9982-0.9964i -1.0155+0.3626i
[4,]  0+0i  0.0000+0.0000i 0.0000+0.0000i  3.0023-3.9998i

Q:
                [,1]             [,2]             [,3]             [,4]
[1,] -0.5312-0.6581i -0.0799-0.3774i -0.0935-0.2736i  0.1869-0.1321i
[2,]  0.2474-0.1769i -0.4108-0.4021i -0.4015+0.6010i -0.0713+0.2225i
[3,]  0.1874-0.2637i -0.0937+0.6241i -0.5752-0.0389i  0.2581-0.3132i
[4,] -0.1909+0.2262i  0.3457-0.0537i -0.1537+0.1951i  0.7668+0.3747i

> ### http://www.nag.com/lapack-ex/node89.html
> (ret <- qz(exA2$A))
W:
[1]  0.7995+0.0000i -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i

T:
       [,1]      [,2]     [,3]     [,4]
[1,] 0.7995  0.006037 -0.11445 -0.03357
[2,] 0.0000 -0.099412 -0.64834 -0.20258
[3,] 0.0000  0.247764 -0.09941 -0.34742
[4,] 0.0000  0.000000  0.00000 -0.10066

Q:
         [,1]    [,2]    [,3]    [,4]
[1,] -0.65509 -0.3450 -0.1037  0.6641
```

```
[2,] -0.52363  0.6141  0.5807 -0.1068
[3,]  0.53622  0.2935  0.3073  0.7293
[4,] -0.09561  0.6463 -0.7467  0.1249
```

QZ demo ex3_ordqz

```
> demo(ex3_ordqz, 'QZ')


        demo(ex3_ordqz)
        ---- ~~~~~~~~~

Type  <Return>   to start :

> # Reordering eigenvalues
> library(QZ, quiet = TRUE)

> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exAB1$A, exAB1$B, select = select))
ALPHA:
[1] 19.033-57.099i 17.897-22.371i 18.175-45.437i  8.757- 2.919i

BETA:
[1] 6.344+0i 4.474+0i 9.087+0i 2.919+0i

S:
            [,1]          [,2]              [,3]             [,4]
[1,] 19.03-57.1i  0.07-93.12i -128.250- 6.366i -98.392+ 9.509i
[2,]  0.00+ 0.0i 17.90-22.37i    0.581- 4.575i   6.972+17.755i
[3,]  0.00+ 0.0i  0.00+ 0.00i   18.175-45.437i -19.992- 6.063i
[4,]  0.00+ 0.0i  0.00+ 0.00i    0.000+ 0.000i   8.757- 2.919i

T:
         [,1]          [,2]            [,3]            [,4]
[1,] 6.344+0i 1.427-1.821i -4.137-6.323i -1.783-1.262i
[2,] 0.000+0i 4.474+0.000i -0.003-3.720i -2.992-0.076i
[3,] 0.000+0i 0.000+0.000i  9.087+0.000i -0.777-1.003i
[4,] 0.000+0i 0.000+0.000i  0.000+0.000i  2.919+0.000i

Q:
                [,1]              [,2]              [,3]                [,4]
[1,] -0.3347+0.7387i  0.0511-0.3524i -0.2997+0.3302i  0.08899-0.09359i
[2,] -0.1277+0.2493i  0.3749+0.5365i  0.2504+0.0137i -0.39709-0.52213i
[3,] -0.3557+0.0396i -0.4717+0.2407i  0.0591+0.2199i -0.56045+0.47485i
[4,] -0.0126-0.3682i  0.4020-0.0522i  0.1201+0.8198i  0.04567+0.10657i

Z:
                [,1]              [,2]                [,3]
                   [,4]
[1,] -0.9240-0.1977i  0.2234+0.1906i -0.08922-0.09991i
   0.0338+0.04268i
[2,] -0.1716+0.0793i -0.5288+0.0772i  0.27684-0.00803i
```

```
   0.3880-0.67191i
[3,] -0.0793-0.1716i  -0.1722-0.6151i  -0.57435-0.20679i
    -0.2753-0.32832i
[4,]  0.1716-0.0793i   0.3215+0.3418i  -0.58658+0.43433i
   0.3229-0.32726i

> ### http://www.nag.com/lapack-ex/node119.html
> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exAB2$A, exAB2$B, select = select))
ALPHA:
[1] 3.801+0.000i 9.203+0.000i 0.857+1.143i 0.857-1.143i

BETA:
[1] 1.9005 2.3008 0.2857 0.2857

S:
      [,1]     [,2]     [,3]     [,4]
[1,] 3.801  -69.451  50.3135  -43.288
[2,] 0.000    9.203  -0.2001    5.988
[3,] 0.000    0.000   1.4279    4.445
[4,] 0.000    0.000   0.9019   -1.196

T:
     [,1]    [,2]    [,3]    [,4]
[1,]  1.9 -10.228  0.8658  -5.2134
[2,]  0.0   2.301  0.7915   0.4262
[3,]  0.0   0.000  0.8101   0.0000
[4,]  0.0   0.000  0.0000  -0.2823

Q:
       [,1]     [,2]      [,3]     [,4]
[1,] 0.4642  0.78862  0.29148 -0.2786
[2,] 0.5002 -0.59864  0.56379 -0.2713
[3,] 0.5002  0.01541 -0.01074  0.8657
[4,] 0.5331 -0.13952 -0.77270 -0.3151

Z:
         [,1]      [,2]      [,3]      [,4]
[1,] 0.996056 -0.00140  0.08868 -0.002602
[2,] 0.005692 -0.04037 -0.09376 -0.994760
[3,] 0.062609  0.71938 -0.69084  0.036273
[4,] 0.062609 -0.69344 -0.71140  0.095554

> (ret <- ordqz(exAB2$A, exAB2$B, keyword = "ref"))
ALPHA:
[1] 3.801+0.000i 9.203+0.000i 0.857+1.143i 0.857-1.143i

BETA:
[1] 1.9005 2.3008 0.2857 0.2857

S:
      [,1]     [,2]     [,3]     [,4]
```

```
[1,] 3.801 -69.451 50.3135 -43.288
[2,] 0.000   9.203 -0.2001   5.988
[3,] 0.000   0.000  1.4279   4.445
[4,] 0.000   0.000  0.9019  -1.196

T:
     [,1]    [,2]   [,3]    [,4]
[1,]  1.9 -10.228 0.8658 -5.2134
[2,]  0.0   2.301 0.7915  0.4262
[3,]  0.0   0.000 0.8101  0.0000
[4,]  0.0   0.000 0.0000 -0.2823

Q:
       [,1]     [,2]     [,3]    [,4]
[1,] 0.4642  0.78862  0.29148 -0.2786
[2,] 0.5002 -0.59864  0.56379 -0.2713
[3,] 0.5002  0.01541 -0.01074  0.8657
[4,] 0.5331 -0.13952 -0.77270 -0.3151

Z:
         [,1]     [,2]     [,3]      [,4]
[1,] 0.996056 -0.00140  0.08868 -0.002602
[2,] 0.005692 -0.04037 -0.09376 -0.994760
[3,] 0.062609  0.71938 -0.69084  0.036273
[4,] 0.062609 -0.69344 -0.71140  0.095554

> (ret <- ordqz(exAB2$A, exAB2$B, keyword = "cef"))
ALPHA:
[1] 0.8571+1.143i 0.8571-1.143i 0.6172+0.000i 4.0000+0.000i

BETA:
[1] 0.2857 0.2857 0.3086 1.0000

S:
        [,1]   [,2]     [,3]    [,4]
[1,] -38.566 41.488  37.2809  65.427
[2,]   6.827 -5.244 -12.9545 -15.482
[3,]   0.000  0.000   0.6172   3.252
[4,]   0.000  0.000   0.0000   4.000

T:
       [,1]   [,2]    [,3]   [,4]
[1,] -3.368 0.0000  4.9228  9.696
[2,]  0.000 0.9621 -1.1839 -2.988
[3,]  0.000 0.0000  0.3086  1.027
[4,]  0.000 0.0000  0.0000  1.000

Q:
        [,1]     [,2]    [,3]       [,4]
[1,] -0.5521 -0.67876  0.4842 -5.145e-17
[2,] -0.5106  0.06994 -0.4842 -7.071e-01
[3,] -0.5106  0.06994 -0.4842  7.071e-01
[4,] -0.4169  0.72767  0.5447 -2.613e-16
```

```
Z:
        [,1]    [,2]       [,3]       [,4]
[1,]  0.8775 0.43756  1.961e-01  0.000e+00
[2,]  0.1755 0.08751 -9.806e-01  5.145e-17
[3,] -0.3155 0.63281 -2.387e-15  7.071e-01
[4,] -0.3155 0.63281 -2.498e-15 -7.071e-01

> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exA1$A, select = select))
W:
[1] -6.000-7.000i  3.002-4.000i -5.000+2.006i  7.998-0.996i

T:
      [,1]             [,2]             [,3]             [,4]
[1,] -6-7i 0.3254-0.8854i  0.5349-0.0829i  0.0083+0.4285i
[2,]  0+0i 3.0023-3.9998i  0.1669+0.2948i -0.2477-1.0389i
[3,]  0+0i 0.0000+0.0000i -5.0000+2.0060i -0.5188-0.4792i
[4,]  0+0i 0.0000+0.0000i  0.0000+0.0000i  7.9982-0.9964i

Q:
                [,1]             [,2]             [,3]             [,4]
[1,] -0.5312-0.6581i -0.0184-0.2122i -0.3775+0.0311i  0.3003+0.0754i
[2,]  0.2474-0.1769i  0.2150+0.2457i -0.4610+0.3622i -0.2198-0.6395i
[3,]  0.1874-0.2637i -0.0469-0.2699i  0.6166+0.1728i  0.4350-0.4701i
[4,] -0.1909+0.2262i  0.8352-0.2747i -0.0033-0.3207i  0.0869-0.1703i

> ### http://www.nag.com/lapack-ex/node89.html
> select <- c(TRUE, FALSE, FALSE, TRUE)

> (ret <- qz(exA2$A, select = select))
W:
[1]  0.7995+0.0000i -0.1007+0.0000i -0.0994+0.4008i -0.0994-0.4008i

T:
       [,1]      [,2]     [,3]     [,4]
[1,] 0.7995 -0.005914 -0.07508 -0.09268
[2,] 0.0000 -0.100657  0.39367  0.35692
[3,] 0.0000  0.000000 -0.09941 -0.51282
[4,] 0.0000  0.000000  0.31324 -0.09941

Q:
         [,1]    [,2]     [,3]     [,4]
[1,] -0.65509 -0.1210 -0.50323  0.55043
[2,] -0.52363 -0.3286  0.78570  0.02287
[3,]  0.53622 -0.5974  0.09038  0.58945
[4,] -0.09561 -0.7215 -0.34825 -0.59081

> (ret <- ordqz(exA2$A, keyword = "lhp"))
W:
[1] -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i  0.7995+0.0000i
```

```
T:
          [,1]     [,2]    [,3]     [,4]
[1,] -0.09941  0.24919  0.3491 0.089393
[2,] -0.64462 -0.09941  0.2049 0.090443
[3,]  0.00000  0.00000 -0.1007 0.009467
[4,]  0.00000  0.00000  0.0000 0.799482

Q:
         [,1]    [,2]    [,3]     [,4]
[1,] -0.1733 -0.3607 -0.6707 -0.62447
[2,]  0.5173  0.6024  0.1005 -0.59949
[3,]  0.3629  0.3067 -0.7241  0.49992
[4,] -0.7554  0.6426 -0.1252 -0.02709

> (ret <- ordqz(exA2$A, keyword = "rhp"))
W:
[1]  0.7995+0.0000i -0.0994+0.4008i -0.0994-0.4008i -0.1007+0.0000i

T:
       [,1]      [,2]     [,3]     [,4]
[1,] 0.7995  0.006037 -0.11445 -0.03357
[2,] 0.0000 -0.099412 -0.64834 -0.20258
[3,] 0.0000  0.247764 -0.09941 -0.34742
[4,] 0.0000  0.000000  0.00000 -0.10066

Q:
          [,1]    [,2]    [,3]    [,4]
[1,] -0.65509 -0.3450 -0.1037  0.6641
[2,] -0.52363  0.6141  0.5807 -0.1068
[3,]  0.53622  0.2935  0.3073  0.7293
[4,] -0.09561  0.6463 -0.7467  0.1249

> (ret <- ordqz(exA2$A, keyword = "ref"))
W:
[1]  0.7995+0.0000i -0.1007+0.0000i -0.0994+0.4008i -0.0994-0.4008i

T:
       [,1]      [,2]     [,3]     [,4]
[1,] 0.7995 -0.005914 -0.07508 -0.09268
[2,] 0.0000 -0.100657  0.39367  0.35692
[3,] 0.0000  0.000000 -0.09941 -0.51282
[4,] 0.0000  0.000000  0.31324 -0.09941

Q:
          [,1]    [,2]     [,3]     [,4]
[1,] -0.65509 -0.1210 -0.50323  0.55043
[2,] -0.52363 -0.3286  0.78570  0.02287
[3,]  0.53622 -0.5974  0.09038  0.58945
[4,] -0.09561 -0.7215 -0.34825 -0.59081

> (ret <- ordqz(exA2$A, keyword = "cef"))
W:
[1] -0.0994+0.4008i -0.0994-0.4008i  0.7995+0.0000i -0.1007+0.0000i
```

```
T:
          [,1]     [,2]    [,3]      [,4]
[1,] -0.09941  0.24919 0.09306 -0.348147
[2,] -0.64462 -0.09941 0.09259 -0.203889
[3,]  0.00000  0.00000 0.79948  0.009467
[4,]  0.00000  0.00000 0.00000 -0.100657

Q:
         [,1]    [,2]    [,3]    [,4]
[1,] -0.1733 -0.3607 -0.6315  0.6641
[2,]  0.5173  0.6024 -0.5984 -0.1068
[3,]  0.3629  0.3067  0.4923  0.7293
[4,] -0.7554  0.6426 -0.0284  0.1249
```

QZ demo ex4_fda_geigen

```
> demo(ex4_fda_geigen, 'QZ')


        demo(ex4_fda_geigen)
        ---- ~~~~~~~~~~~~~~

Type  <Return>   to start :

> library(QZ, quiet = TRUE)

> ### Generate Data
> set.seed(123)

> X <- matrix(rnorm(500), nrow = 25)

> X <- t(X) %*% X

> A <- X[1:8, 9:20]

> B <- X[1:8, 1:8]

> C <- X[9:20, 9:20]

> ### Perform generalized eigenanalysis
> ret.qz <- fda.geigen(A, B, C)

> ret.fda <- fda::geigen(A, B, C)

> ### Verify
> round(abs(ret.qz$values - ret.fda$values))
[1] 0 0 0 0 0 0 0 0

> round(abs(ret.qz$Lmat - ret.fda$Lmat))
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    0    0    0    0    0    0    0    0
[2,]    0    0    0    0    0    0    0    0
```

```
[3,]     0     0     0     0     0     0     0     0
[4,]     0     0     0     0     0     0     0     0
[5,]     0     0     0     0     0     0     0     0
[6,]     0     0     0     0     0     0     0     0
[7,]     0     0     0     0     0     0     0     0
[8,]     0     0     0     0     0     0     0     0

> round(abs(ret.qz$Mmat - ret.fda$Mmat))
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
 [1,]     0     0     0     0     0     0     0     0
 [2,]     0     0     0     0     0     0     0     0
 [3,]     0     0     0     0     0     0     0     0
 [4,]     0     0     0     0     0     0     0     0
 [5,]     0     0     0     0     0     0     0     0
 [6,]     0     0     0     0     0     0     0     0
 [7,]     0     0     0     0     0     0     0     0
 [8,]     0     0     0     0     0     0     0     0
 [9,]     0     0     0     0     0     0     0     0
[10,]     0     0     0     0     0     0     0     0
[11,]     0     0     0     0     0     0     0     0
[12,]     0     0     0     0     0     0     0     0
```

References

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Ramsay J, Wickham H, Graves S, Hooker G (2013). "fda: Functional Data Analysis." R Package, URL http://cran.r-project.org/package=fda.