# Package 'NPP'

August 21, 2019

2 BerNPP\_MCMC

BerNP	PP_MCMC	MCM Prior	Sai	np	ling	g fo	r.	Вe	rn	ои	lli	Pa	ррі	ıla	tio	n	us	ing	3 I	Vo	rm	ali	zec	l I	Por	ve	r
Index																											29
	VaccineData		 	•		٠			•			•	•		•	•		•	•	•		•		•	•	•	28
	SPDData																										
	PoissonNPP_MCM																										
	PHData																										
	NormalNPP_MCM																										
	MultinomialNPP_M	1CMC	 																								19
	ModeDeltaPoisNPF	·	 																								18
	ModeDeltaNormalN	NPP .	 																								16

# **Description**

Conduct posterior sampling for Bernoulli population with normalized power prior. For the power parameter  $\delta$ , a Metropolis-Hastings algorithm with either independence proposal, or a random walk proposal on its logit scale is used. For the model parameter p, Gibbs sampling is used.

# Usage

# Arguments

Data.Cur	a non-negative integer vector of two elements: c(number of success, number of failure) in the current data.
Data.Hist	a non-negative integer vector of two elements: c(number of success, number of failure) in the historical data.
CompStat	a list of four elements that represents the "compatibility(sufficient) statistics" for $p$ . Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data.Cur and Data.Hist will be ignored. Note: in Bernoulli population providing CompStat is equivalent to provide the data summary as in Data.Cur and Data.Cur.  n0 is the number of trials in the historical data.  y0 is the number of successes in the historical data.  y1 is the number of successes in the current data.

BerNPP\_MCMC 3

prior a list of the hyperparameters in the prior for both p and  $\delta$ .

p. alpha is the hyperparameter  $\alpha$  in the prior distribution  $Beta(\alpha,\beta)$  for p. p. beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha,\beta)$  for p. delta.alpha is the hyperparameter  $\alpha$  in the prior distribution  $Beta(\alpha,\beta)$  for

 $\delta$ .

delta. beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha, \beta)$  for  $\delta$ .

MCMCmethod sampling method for  $\delta$  in MCMC. It can be either 'IND' for independence pro-

posal; or 'RW' for random walk proposal on logit scale.

rw.logit.delta the stepsize(variance of the normal distribution) for the random walk proposal

of logit  $\delta$ . Only applicable if MCMCmethod = 'RW'.

ind.delta.alpha

specifies the first parameter  $\alpha$  when independent proposal  $Beta(\alpha, \beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

ind.delta.beta specifies the first parameter  $\beta$  when independent proposal  $Beta(\alpha,\beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

nsample specifies the number of posterior samples in the output.

control.mcmc a list of three elements used in posterior sampling.

delta.ini is the initial value of  $\delta$  in MCMC sampling.

burnin is the number of burn-ins. The output will only show MCMC samples

after bunrin.

thin is the thinning parameter in MCMC sampling.

#### **Details**

The outputs include posteriors of the model parameter(s) and power parameter, acceptance rate in sampling  $\delta$ , and the deviance information criteria.

#### Value

A list of class "NPP" with four elements:

p posterior of the model parameter p. delta posterior of the power parameter  $\delta$ .

acceptance the acceptance rate in MCMC sampling for  $\delta$  using Metropolis-Hastings algo-

rithm.

DIC the deviance information criteria for model diagnostics.

# Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

4 LaplacelogC

#### See Also

MultinomialNPP\_MCMC; NormalNPP\_MCMC; PoissonNPP\_MCMC

#### **Examples**

```
\label{eq:bernPP_MCMC} \begin{aligned} \text{BerNPP\_MCMC}(\text{Data.Cur} = \text{c}(493, \ 473), \ \text{Data.Hist} = \text{c}(680, \ 669), \\ \text{prior} &= \text{list}(\text{p.alpha} = 0.5, \ \text{p.beta} = 0.5, \ \text{delta.alpha} = 1, \ \text{delta.beta} = 1), \\ \text{MCMCmethod} &= \text{'RW'}, \ \text{rw.logit.delta} = 1, \ \text{nsample} = 5000, \\ \text{control.mcmc} &= \text{list}(\text{delta.ini} = \text{NULL}, \ \text{burnin} = 2000, \ \text{thin} = 5)) \end{aligned}
```

LaplacelogC

A Function to Calculate  $logC(\delta)$  Based on Laplace Approximation

# **Description**

The function assumes that the prior of the model parameters is very flat that had very minor impact on the shape of the power prior (posterior based on the D0).

#### Usage

```
LaplacelogC(delta, loglikmle, detHessian, ntheta)
```

#### **Arguments**

delta	the power parameter between 0 and 1. The function returns $logC(\delta)$
loglikmle	a scalar; the loglikelihood of the historical data evaluated at the maximum likelihood estimates based on the historical data
detHessian	determinant of the Hessian matrix evaluated at the loglikelihood function with respect to the maximum likelihood estimates based on the historical data
ntheta	an positive integer indicating number of parameters in the model

#### Value

 $logC(\delta)$  based on the Laplace approximation. Can be used for the posterior sampling in the normalized power prior.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

LMNPP\_MCMC 5

#### See Also

logCknot

LMNPP\_MCMC

MCMC Sampling for Normal Linear Model using Normalized Power Prior

# Description

Conduct posterior sampling for normal linear model with normalized power prior. For the power parameter  $\delta$ , a Metropolis-Hastings algorithm with either independence proposal, or a random walk proposal on its logit scale is used. For the regression parameter  $\beta$  and  $\sigma^2$ , Gibbs sampling is used.

# Usage

# Arguments

8	
y.Cur	a vector of individual level of the response y in current data.
y.Hist	a vector of individual level of the response y in historical data.
x.Cur	a vector or matrix or data frame of covariate observed in the current data. If more than 1 covariate available, the number of rows is equal to the number of observations.
x.Hist	a vector or matrix or data frame of covariate observed in the historical data. If more than 1 covariate available, the number of rows is equal to the number of observations.
prior	a list of the hyperparameters in the prior for model parameters $(\beta, \sigma^2)$ and $\delta$ . The form of the prior for model parameter $(\beta, \sigma^2)$ is in the section "Details". a a positive hyperparameter for prior on model parameters. It is the power $a$ in formula $(1/\sigma^2)^a$ ; See details.
	b equals 0 if a flat prior is used for $\beta$ . Equals 1 if a normal prior is used for $\beta$ ; See details.
	mu0 a vector of the mean for prior $\beta   \sigma^2$ . Only applicable if b = 1.
	Rinv inverse of the matrix $R$ . The covariance matrix of the prior for $\beta \sigma^2$ is $\sigma^2R^{-1}$ .
	delta.alpha is the hyperparameter $\alpha$ in the prior distribution $Beta(\alpha,\beta)$ for

delta. beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha, \beta)$  for  $\delta$ .

6 LMNPP\_MCMC

MCMCmethod sampling method for  $\delta$  in MCMC. It can be either 'IND' for independence pro-

posal; or 'RW' for random walk proposal on logit scale.

rw.logit.delta the stepsize(variance of the normal distribution) for the random walk proposal

of logit  $\delta$ . Only applicable if MCMCmethod = 'RW'.

ind.delta.alpha

specifies the first parameter  $\alpha$  when independent proposal  $Beta(\alpha, \beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

ind.delta.beta specifies the first parameter  $\beta$  when independent proposal  $Beta(\alpha,\beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

nsample specifies the number of posterior samples in the output. control.mcmc a list of three elements used in posterior sampling.

delta. ini is the initial value of  $\delta$  in MCMC sampling.

burnin is the number of burn-ins. The output will only show MCMC samples

after bunrin.

thin is the thinning parameter in MCMC sampling.

#### **Details**

If b=1, prior for  $(\beta,\sigma)$  is  $(1/\sigma^2)^a*N(mu0,\sigma^2R^{-1})$ , which includes the g-prior. If b=0, prior for  $(\beta,\sigma)$  is  $(1/\sigma^2)^a$ . The outputs include posteriors of the model parameter(s) and power parameter, acceptance rate when sampling  $\delta$ , and the deviance information criteria.

#### Value

A list of class "NPP" with five elements:

beta posterior of the model parameter  $\beta$  in vector or matrix form.

sigmasq posterior of the model parameter  $\sigma^2$ . delta posterior of the power parameter  $\delta$ .

acceptance the acceptance rate in MCMC sampling for  $\delta$  using Metropolis-Hastings algo-

rithm.

DIC the deviance information criteria for model diagnostics.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

Berger, J.O. and Bernardo, J.M. (1992). On the development of reference priors. *Bayesian Statistics* 4: Proceedings of the Fourth Valencia International Meeting, Bernardo, J.M, Berger, J.O., Dawid, A.P. and Smith, A.F.M. eds., 35-60, Clarendon Press:Oxford.

Jeffreys, H. (1946). An Invariant Form for the Prior Probability in Estimation Problems. *Proceedings of the Royal Statistical Society of London, Series A* 186:453-461.

logCdelta 7

#### See Also

BerNPP\_MCMC; MultinomialNPP\_MCMC; PoissonNPP\_MCMC; NormalNPP\_MCMC

#### **Examples**

logCdelta

A Function to Interpolate  $logC(\delta)$  Based on Its Values on Selected Knots

# **Description**

The function returns the interpolated value (a scalar) of  $logC(\delta)$  based on its results on selected knots, given input vector of  $\delta$ .

#### Usage

```
logCdelta(delta, deltaknot, lCknot)
```

#### **Arguments**

delta a scalar of the input value of  $\delta$ .

deltaknot a vector of the knots for  $\delta$ . It should be selected before conduct the sampling.

1Cknot a vector of the values  $logC(\delta)$  on selected knots, coming from the function

logCknot.

#### Value

A sequence of the values,  $logC(\delta)$  on selected knots.

# Author(s)

Zifei Han <hanzifei1@gmail.com>

8 logCknot

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

loglikNormD0; loglikBerD0; logCknot

logCknot

A Function to Calculate  $logC(\delta)$  on Selected Knots

# Description

The function returns a sequence of the values,  $logC(\delta)$  on selected knots, given input vector of  $\delta$ .

#### Usage

```
logCknot(deltaknot, llikf0)
```

# Arguments

deltaknot a vector of the knots for  $\delta$ . It should be selected before conduct the sampling.

llikf0 a matrix of the log-likelihoods of class "npp".

#### Value

A sequence of the values,  $logC(\delta)$  on selected knots.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

# References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

# See Also

loglikNormD0; loglikBerD0; logCdelta

loglikBerD0

loglikBerD0	A Function to Calculate Log-likelihood of the Historical Data, Given
	Matrix-valued Parameters, for Bernoulli Population

#### **Description**

The function returns a matrix of class "npp", each element is a log-likelihood of the historical data. It is an intermediate step to calculate the "normalizing constant"  $C(\delta)$  in the normalized power prior, for the purpose of providing a flexible implementation. Users can specify their own likelihood function of the same class following this structure.

# Usage

```
loglikBerD0(D0, thetalist, ntheta = 1)
```

#### **Arguments**

D0 a vector of each observation(binary) in historical data.

thetalist a list of parameter values. The number of elements is equal to ntheta. Each

element is a matrix. The sample should come from the posterior of the powered likelihood for historical data, with each column corresponds to a distinct value of the power parameter  $\delta$  (the corresponding power parameter increases from left to right). The number of rows is the number of Monte Carlo samples for each  $\delta$  fixed. The number of columns is the number of selected knots (number

of distinct  $\delta$ ).

ntheta a positive integer indicating number of parameters to be estimated in the model.

Default is 1 for Bernoulli.

# Value

A numeric matrix of log-likelihood, for the historical data given the matrix(or array)-valued parameters.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

loglikNormD0; logCknot; logCdelta

10 loglikNormD0

loglikNormD0	A Function to Calculate Log-likelihood of the Historical Data, Given
	Array-valued Parameters, for Normal Population

# **Description**

The function returns a matrix of class "npp", each element is a log-likelihood of the historical data. It is an intermediate step to calculate the "normalizing constant"  $C(\delta)$  in the normalized power prior, for the purpose of providing a flexible implementation. Users can specify their own likelihood function of the same class following this structure.

#### Usage

```
loglikNormD0(D0, thetalist, ntheta = 2)
```

# **Arguments**

D0 a vector of each observation in historical data.

thetalist a list of parameter values. The number of elements is equal to ntheta. Each

element is a matrix. The sample should come from the posterior of the powered likelihood for historical data, with each column corresponds to a distinct value of the power parameter  $\delta$  (the corresponding power parameter increases from left to right). The number of rows is the number of Monte Carlo samples for each  $\delta$  fixed. The number of columns is the number of selected knots (number

of distinct  $\delta$ ).

ntheta a positive integer indicating number of parameters to be estimated in the model.

#### Value

A numeric matrix of log-likelihood, for the historical data given the matrix(or array)-valued parameters.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

```
loglikBerD0; logCknot; logCdelta
```

ModeDeltaBerNPP 11

ModeDeltaBerNPP	Calculate Posterior Mode of the Power Parameter in Normalized
	Power Prior with Grid Search, Bernoulli Population

# Description

The function returns the posterior mode of the power parameter  $\delta$  in Bernoulli population. It calculates the log of the posterior density (up to a normalizing constant), and conduct a grid search to find the approximate mode.

# Usage

```
ModeDeltaBerNPP(Data.Cur, Data.Hist,
                CompStat = list(n0 = NULL, y0 = NULL, n1 = NULL, y1 = NULL),
                npoints = 1000,
                prior = list(p.alpha = 1, p.beta = 1,
                             delta.alpha = 1, delta.beta = 1))
```

#### Ar

rguments	
Data.Cur	a non-negative integer vector of two elements: c(number of success, number of failure) in the current data.
Data.Hist	a non-negative integer vector of two elements: c(number of success, number of failure) in the historical data.
CompStat	a list of four elements that represents the "compatibility(sufficient) statistics" for $p$ . Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data.Cur and Data.Hist will be ignored. Note: in Bernoulli population providing CompStat is equivalent to provide the data summary as in Data.Cur and Data.Cur.  n0 is the number of trials in the historical data.  y0 is the number of successes in the historical data.  n1 is the number of successes in the current data.
npoints	is a non-negative integer scalar indicating number of points on a regular spaced grid between [0, 1], where we calculate the log of the posterior and search for the mode.
prior	a list of the hyperparameters in the prior for both $p$ and $\delta$ . p.alpha is the hyperparameter $\alpha$ in the prior distribution $Beta(\alpha,\beta)$ for $p$ . p.beta is the hyperparameter $\beta$ in the prior distribution $Beta(\alpha,\beta)$ for $p$ . delta.alpha is the hyperparameter $\alpha$ in the prior distribution $Beta(\alpha,\beta)$ for

delta.beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha, \beta)$  for  $\delta$ .

# **Details**

See example.

 $\delta$ .

12 ModeDeltaLMNPP

#### Value

A numeric value between 0 and 1.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

ModeDeltaNormalNPP; ModeDeltaPoisNPP; ModeDeltaMultinomialNPP

# Examples

```
ModeDeltaBerNPP(Data.Cur = c(100, 40), Data.Hist = c(100, 40), npoints = 1000, prior = list(p.alpha = 1, p.beta = 1, delta.alpha = 1, delta.beta = 1))

ModeDeltaBerNPP(Data.Cur = c(100, 40), Data.Hist = c(100, 35), npoints = 1000, prior = list(p.alpha = 1, p.beta = 1, delta.alpha = 1, delta.beta = 1))

ModeDeltaBerNPP(Data.Cur = c(100, 40), Data.Hist = c(100, 50), npoints = 1000, prior = list(p.alpha = 1, p.beta = 1, delta.alpha = 1, delta.beta = 1))
```

ModeDeltaLMNPP

Calculate Posterior Mode of the Power Parameter in Normalized Power Prior with Grid Search, Normal Linear Model

# Description

The function returns the posterior mode of the power parameter  $\delta$  in normal linear model. It calculates the log of the posterior density (up to a normalizing constant), and conduct a grid search to find the approximate mode.

# Usage

ModeDeltaLMNPP 13

a vector of individual level of the response y in current data.

#### **Arguments**

y.Cur

y.cui	a vector of marvidual level of the response y in earliest data.
y.Hist	a vector of individual level of the response y in historical data.
x.Cur	a vector or matrix or data frame of covariate observed in the current data. If more than 1 covariate available, the number of rows is equal to the number of observations.
x.Hist	a vector or matrix or data frame of covariate observed in the historical data. If more than 1 covariate available, the number of rows is equal to the number of observations.
npoints	is a non-negative integer scalar indicating number of points on a regular spaced grid between [0, 1], where we calculate the log of the posterior and search for the mode.
prior	a list of the hyperparameters in the prior for model parameters $(\beta, \sigma^2)$ and $\delta$ . The form of the prior for model parameter $(\beta, \sigma^2)$ is in the section "Details".
	a a positive hyperparameter for prior on model parameters. It is the power $a$ in formula $(1/\sigma^2)^a$ ; See details.
	b equals 0 if a flat prior is used for $\beta$ . Equals 1 if a normal prior is used for $\beta$ ; See details.
	mu0 a vector of the mean for prior $\beta   \sigma^2$ . Only applicable if b = 1.
	Rinv inverse of the matrix $R$ . The covariance matrix of the prior for $\beta \sigma^2$ is $\sigma^2R^{-1}$ .
	delta.alpha is the hyperparameter $\alpha$ in the prior distribution $Beta(\alpha,\beta)$ for

#### **Details**

If b=1, prior for  $(\beta,\sigma)$  is  $(1/\sigma^2)^a*N(mu0,\sigma^2R^{-1})$ , which includes the g-prior. If b=0, prior for  $(\beta,\sigma)$  is  $(1/\sigma^2)^a$ . The outputs include posteriors of the model parameter(s) and power parameter, acceptance rate when sampling  $\delta$ , and the deviance information criteria.

delta. beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha,\beta)$  for  $\delta$ .

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

Berger, J.O. and Bernardo, J.M. (1992). On the development of reference priors. *Bayesian Statistics* 4: Proceedings of the Fourth Valencia International Meeting, Bernardo, J.M, Berger, J.O., Dawid, A.P. and Smith, A.F.M. eds., 35-60, Clarendon Press:Oxford.

Jeffreys, H. (1946). An Invariant Form for the Prior Probability in Estimation Problems. *Proceedings of the Royal Statistical Society of London, Series A 186:453-461*.

#### See Also

ModeDeltaBerNPP; ModeDeltaNormalNPP; ModeDeltaMultinomialNPP; ModeDeltaNormalNPP

#### ModeDeltaMultinomialNPP

Calculate Posterior Mode of the Power Parameter in Normalized Power Prior with Grid Search, Multinomial Population

#### **Description**

The function returns the posterior mode of the power parameter  $\delta$  in multinomial population. It calculates the log of the posterior density (up to a normalizing constant), and conduct a grid search to find the approximate mode.

# Usage

```
ModeDeltaMultinomialNPP(Data.Cur, Data.Hist, CompStat = list(n0 = NULL, n1 = NULL),
                  npoints = 1000, prior = list(theta.dir.alpha = c(0.5, 0.5, 0.5),
                        delta.alpha = 1, delta.beta = 1))
```

#### **Arguments**

Data.Cur	a non-negative integer	vector of $K$ elements:	c(number of	success in group	1,

number of success in group 2, ..., number of success in group K) in the current

data.

Data.Hist a non-negative integer vector of K elements: c(number of success in group 1,

number of success in group 2, ..., number of success in group K) in the historical

data.

CompStat a list of two elements that represents the "compatibility(sufficient) statistics" for

> $\theta$ . Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data. Cur and Data. Hist will be ignored. Note: in multinomial case providing CompStat is equivalent to provide the data summary

as in Data. Cur and Data. Cur.

n0 is a non-negative integer vector of K elements for compatible statistics in historical data: c(number of success in group 1, number of success in group 2,

..., number of success in group K).

n1 is a non-negative integer vector of K elements for compatible statistics in current data: c(number of success in group 1, number of success in group 2, ...,

number of success in group K).

is a non-negative integer scalar indicating number of points on a regular spaced npoints grid between [0, 1], where we calculate the log of the posterior and search for

the mode.

prior

a list of the hyperparameters in the prior for both p and  $\delta$ .

theta.dir is a vector of K elements of the hyperparameter  $\alpha$  in the prior distribution  $Dir(\alpha[1], \alpha[2], ..., \alpha[K])$  for  $\theta$ .

delta. alpha a scalar, the hyperparameter  $\alpha$  in the prior distribution  $Beta(\alpha,\beta)$  for  $\delta$ .

delta.beta a scalar, the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha,\beta)$  for  $\delta$ .

#### **Details**

See example.

#### Value

A numeric value between 0 and 1.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

ModeDeltaBerNPP; ModeDeltaNormalNPP; ModeDeltaPoisNPP

#### **Examples**

```
\label{eq:modeDeltaMultinomialNPP} $$ \mbox{ModeDeltaMultinomialNPP(CompStat = list(n0 = c(25,25,25,25), n1 = c(25,25,25,25)), prior = list(theta.dir.alpha = c(0.5, 0.5, 0.5, 0.5), delta.alpha = 1, delta.beta = 1)) $$ \mbox{ModeDeltaMultinomialNPP(CompStat = list(n0 = c(22,25,28,25), n1 = c(25,22,25,28)), prior = list(theta.dir.alpha = c(0.5, 0.5, 0.5, 0.5), delta.alpha = 1, delta.beta = 1)) $$ \mbox{ModeDeltaMultinomialNPP(CompStat = list(n0 = c(15,25,30,30), n1 = c(25,25,25,25)), prior = list(theta.dir.alpha = c(0.5, 0.5, 0.5, 0.5), delta.alpha = 1, delta.beta = 1)) $$ \mbox{delta.alpha = 1, delta.beta = 1)} $$
```

16 ModeDeltaNormalNPP

ModeDeltaNormalNPP

Calculate Posterior Mode of the Power Parameter in Normalized Power Prior with Grid Search, Normal Population

#### **Description**

The function returns the posterior mode of the power parameter  $\delta$  in multinomial population. It calculates the log of the posterior density (up to a normalizing constant), and conduct a grid search to find the approximate mode.

#### Usage

#### **Arguments**

Data.Cur a vector of individual level current data.

Data.Hist a vector of individual level historical data.

CompStat

a list of six elements(scalar) that represents the "compatibility(sufficient) statistics" for model parameters. Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data.Cur and Data.Hist will be ignored.

no is the sample size of historical data.

mean0 is the sample mean of the historical data. var0 is the sample variance of the historical data.

n1 is the sample size of current data.

mean1 is the sample mean of the current data. var1 is the sample variance of the current data.

npoints

is a non-negative integer scalar indicating number of points on a regular spaced grid between [0, 1], where we calculate the log of the posterior and search for the mode.

prior

a list of the hyperparameters in the prior for both  $(\mu,\sigma^2)$  and  $\delta$ . The form of the prior for model parameter  $(\mu,\sigma^2)$  is  $(1/\sigma^2)^a$ . When a=1 it corresponds to the reference prior, and when a=1.5 it corresponds to the Jeffrey's prior.

a is the power a in formula  $(1/\sigma^2)^a$ , the prior for  $(\mu, \sigma^2)$  jointly.

delta.alpha is the hyperparameter  $\alpha$  in the prior distribution  $Beta(\alpha,\beta)$  for  ${}^{\rm c}$ 

0.

delta. beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha, \beta)$  for  $\delta$ .

ModeDeltaNormalNPP 17

#### **Details**

See example.

# Value

A numeric value between 0 and 1.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

Berger, J.O. and Bernardo, J.M. (1992). On the development of reference priors. *Bayesian Statistics* 4: Proceedings of the Fourth Valencia International Meeting, Bernardo, J.M, Berger, J.O., Dawid, A.P. and Smith, A.F.M. eds., 35-60, Clarendon Press:Oxford.

Jeffreys, H. (1946). An Invariant Form for the Prior Probability in Estimation Problems. *Proceedings of the Royal Statistical Society of London, Series A 186:453-461*.

# See Also

ModeDeltaBerNPP; ModeDeltaMultinomialNPP; ModeDeltaPoisNPP

# **Examples**

18 ModeDeltaPoisNPP

Power Prior with Grid Search, Poisson Population	ModeDeltaPoisNPP	Calculate Posterior Mode of the Power Parameter in Normalized Power Prior with Grid Search, Poisson Population
--	------------------	---

# Description

The function returns the posterior mode of the power parameter  $\delta$  in multinomial population. It calculates the log of the posterior density (up to a normalizing constant), and conduct a grid search to find the approximate mode.

# Usage

# **Arguments**

r	rguments					
	Data.Cur	a non-negative integer vector of each observed current data.				
	Data.Hist	a non-negative integer vector of each observed historical data.				
	CompStat	a list of four elements that represents the "compatibility(sufficient) statistics" for $\lambda$ . Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data. Cur and Data. Hist will be ignored. no is the number of observations in the historical data. mean 0 is the sample mean of the historical data.				
		n1 is the number of observations in the current data.				
		mean1 is the sample mean of the current data.				
	npoints	is a non-negative integer scalar indicating number of points on a regular spaced grid between [0, 1], where we calculate the log of the posterior and search for the mode.				
	prior	a list of the hyperparameters in the prior for both $\lambda$ and $\delta$ . A Gamma distribution is used as the prior of $\lambda$ , and a Beta distribution is used as the prior of $\delta$ .				
		lambda . shape is the shape (hyper)parameter in the prior distribution $Gamma(shape, scale)$ for $\lambda$ .				
		lambda . scale is the scale (hyper)parameter in the prior distribution $Gamma(shape, scale)$ for $\lambda$ .				
		delta.alpha is the hyperparameter $\alpha$ in the prior distribution $Beta(\alpha,\beta)$ for $\delta.$				
		delta.beta is the hyperparameter $\beta$ in the prior distribution $Beta(\alpha,\beta)$ for $\delta$ .				

# **Details**

See example.

#### Value

A numeric value between 0 and 1.

# Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

ModeDeltaBerNPP; ModeDeltaNormalNPP; ModeDeltaMultinomialNPP

# Examples

MultinomialNPP\_MCMC

MCMC Sampling for Multinomial Population using Normalized Power Prior

# Description

Conduct posterior sampling for multinomial population with normalized power prior. For the power parameter  $\delta$ , a Metropolis-Hastings algorithm with either independence proposal, or a random walk proposal on its logit scale is used. For the model parameter vector  $\theta$ , Gibbs sampling is used. Assume the prior for model parameter  $\theta$  comes from a Dirichlet distribution.

#### Usage

```
MultinomialNPP_MCMC(Data.Cur = c(10, 10, 10), Data.Hist = c(10, 10, 10),
                    CompStat = list(n0 = NULL, n1 = NULL),
                    prior = list(theta.dir = c(0.5, 0.5, 0.5),
                                 delta.alpha = 1, delta.beta = 1),
                    MCMCmethod = 'IND', rw.logit.delta = 0.1,
                    ind.delta.alpha = 1, ind.delta.beta = 1, nsample = 5000,
                   control.mcmc = list(delta.ini = NULL, burnin = 0, thin = 1))
```

#### **Arguments**

Data.Cur

a non-negative integer vector of K elements: c(number of success in group 1, number of success in group 2, ..., number of success in group K) in the current data.

Data.Hist

a non-negative integer vector of K elements: c(number of success in group 1, number of success in group 2, ..., number of success in group K) in the historical data.

CompStat

a list of two elements that represents the "compatibility(sufficient) statistics" for  $\theta$ . Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data. Cur and Data. Hist will be ignored. Note: in multinomial case providing CompStat is equivalent to provide the data summary as in Data. Cur and Data. Cur.

n0 is a non-negative integer vector of K elements for compatible statistics in historical data: c(number of success in group 1, number of success in group 2, ..., number of success in group K).

n1 is a non-negative integer vector of K elements for compatible statistics in current data: c(number of success in group 1, number of success in group 2, ..., number of success in group K).

prior

a list of the hyperparameters in the prior for both p and  $\delta$ .

theta. dir is a vector of K elements of the hyperparameter  $\alpha$  in the prior distribution  $Dir(\alpha[1], \alpha[2], ..., \alpha[K])$  for  $\theta$ .

delta.alpha a scalar, the hyperparameter  $\alpha$  in the prior distribution  $Beta(\alpha,\beta)$ for  $\delta$ .

delta. beta a scalar, the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha, \beta)$ 

MCMCmethod

sampling method for  $\delta$  in MCMC. It can be either 'IND' for independence proposal; or 'RW' for random walk proposal on logit scale.

rw.logit.delta the stepsize(variance of the normal distribution) for the random walk proposal of logit  $\delta$ . Only applicable if MCMCmethod = 'RW'.

ind.delta.alpha

specifies the first parameter  $\alpha$  when independent proposal  $Beta(\alpha, \beta)$  for  $\delta$  is used. Only applicable if MCMCmethod = 'IND'

ind.delta.beta specifies the first parameter  $\beta$  when independent proposal  $Beta(\alpha,\beta)$  for  $\delta$  is used. Only applicable if MCMCmethod = 'IND'

nsample specifies the number of posterior samples in the output.

control.mcmc a list of three elements used in posterior sampling.

delta. ini is the initial value of  $\delta$  in MCMC sampling.

burnin is the number of burn-ins. The output will only show MCMC samples

after bunrin.

thin is the thinning parameter in MCMC sampling.

#### **Details**

The outputs include posteriors of the model parameter(s) and power parameter, acceptance rate in sampling  $\delta$ , and the deviance information criteria.

#### Value

A list of class "NPP" with four elements:

p posterior of the model parameter  $\theta$ . delta posterior of the power parameter  $\delta$ .

acceptance the acceptance rate in MCMC sampling for  $\delta$  using Metropolis-Hastings algo-

rithm.

DIC the deviance information criteria for model diagnostics.

#### Author(s)

Tianyu Bai <tianyu.bai24@gmail.com> Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

# See Also

```
BerNPP_MCMC; NormalNPP_MCMC; PoissonNPP_MCMC
```

# **Examples**

```
\label{eq:multinomialNPP_MCMC} \begin{split} \text{MultinomialNPP\_MCMC(Data.Cur} &= c(3,11,3,669), \; \text{Data.Hist} = c(9,20,9,473), \\ & \quad \text{prior} = \text{list(theta.dir} = c(1,1,1,1), \\ & \quad \text{delta.alpha} = 1, \; \text{delta.beta} = 1), \\ & \quad \text{MCMCmethod} = '\text{IND'}, \; \text{rw.logit.delta} = 0.1, \\ & \quad \text{ind.delta.alpha} = 1, \; \text{ind.delta.beta} = 1, \; \text{nsample} = 10000, \\ & \quad \text{control.mcmc} = \text{list(delta.ini} = \text{NULL}, \\ & \quad \text{burnin} = 2000, \; \text{thin} = 5)) \end{split}
```

22 NormalNPP\_MCMC

NormalNPP\_MCMC

MCMC Sampling for Normal Population using Normalized Power Prior

#### **Description**

Conduct posterior sampling for normal population with normalized power prior. The initial prior  $\pi(\mu|\sigma^2)$  is a flat prior. For the power parameter  $\delta$ , a Metropolis-Hastings algorithm with either independence proposal, or a random walk proposal on its logit scale is used. For the model parameter  $\mu$  and  $\sigma^2$ , Gibbs sampling is used.

#### Usage

#### **Arguments**

Data.Cur a vector of individual level current data.

Data.Hist a vector of individual level historical data.

CompStat

a list of six elements(scalar) that represents the "compatibility(sufficient) statistics" for model parameters. Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data.Cur and Data.Hist will be ignored.

n0 is the sample size of historical data.

mean0 is the sample mean of the historical data. var0 is the sample variance of the historical data.

n1 is the sample size of current data.

mean1 is the sample mean of the current data. var1 is the sample variance of the current data.

prior

a list of the hyperparameters in the prior for both  $(\mu, \sigma^2)$  and  $\delta$ . The form of the prior for model parameter  $(\mu, \sigma^2)$  is  $(1/\sigma^2)^a$ . When a=1 it corresponds to the reference prior, and when a=1.5 it corresponds to the Jeffrey's prior.

a is the power a in formula  $(1/\sigma^2)^a$ , the prior for  $(\mu, \sigma^2)$  jointly.

delta.alpha is the hyperparameter  $\alpha$  in the prior distribution  $Beta(\alpha,\beta)$  for  $\delta$ .

delta.beta is the hyperparameter  $\beta$  in the prior distribution  $Beta(\alpha, \beta)$  for  $\delta$ .

MCMCmethod

sampling method for  $\delta$  in MCMC. It can be either 'IND' for independence pro-

posal; or 'RW' for random walk proposal on logit scale.

NormalNPP\_MCMC 23

rw.logit.delta the stepsize(variance of the normal distribution) for the random walk proposal

of logit  $\delta$ . Only applicable if MCMCmethod = 'RW'.

ind.delta.alpha

specifies the first parameter  $\alpha$  when independent proposal  $Beta(\alpha, \beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

ind.delta.beta specifies the first parameter  $\beta$  when independent proposal  $Beta(\alpha,\beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

nsample specifies the number of posterior samples in the output.

control.mcmc a list of three elements used in posterior sampling.

delta.ini is the initial value of  $\delta$  in MCMC sampling.

burnin is the number of burn-ins. The output will only show MCMC samples

after bunrin.

thin is the thinning parameter in MCMC sampling.

#### **Details**

The outputs include posteriors of the model parameter(s) and power parameter, acceptance rate in sampling  $\delta$ , and the deviance information criteria.

#### Value

A list of class "NPP" with five elements:

mu posterior of the model parameter  $\mu$ . sigmasq posterior of the model parameter  $\sigma^2$ . delta posterior of the power parameter  $\delta$ .

acceptance the acceptance rate in MCMC sampling for  $\delta$  using Metropolis-Hastings algo-

rithm.

DIC the deviance information criteria for model diagnostics.

#### Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

Berger, J.O. and Bernardo, J.M. (1992). On the development of reference priors. *Bayesian Statistics* 4: *Proceedings of the Fourth Valencia International Meeting, Bernardo, J.M, Berger, J.O., Dawid, A.P. and Smith, A.F.M. eds.*, 35-60, Clarendon Press:Oxford.

Jeffreys, H. (1946). An Invariant Form for the Prior Probability in Estimation Problems. *Proceedings of the Royal Statistical Society of London, Series A 186:453-461*.

24 PHData

### See Also

```
BerNPP_MCMC; MultinomialNPP_MCMC; PoissonNPP_MCMC;
```

# **Examples**

PHData

PH Data on four sites in Virginia

# Description

The dataset is used to assess if there is site impairment. The site impairment is defined as whether the pH values at a site indicate that the site violates a (lower) standard of 6.0 more than 10% of the time.

#### Usage

```
data("PHData")
```

#### **Format**

A data frame with 325 observations on the following 3 variables.

#### **Examples**

```
data(PHData)
```

PoissonNPP\_MCMC 25

PoissonNPP_MCMC	MCMC Sampling for Bernoulli Population using Normalized Power
	Prior

# **Description**

Conduct posterior sampling for Poisson population with normalized power prior. For the power parameter  $\delta$ , a Metropolis-Hastings algorithm with either independence proposal, or a random walk proposal on its logit scale is used. For the model parameter  $\lambda$ , Gibbs sampling is used.

# Usage

#### **Arguments**

guments				
Data.Cur	a non-negative integer vector of each observed current data.			
Data.Hist	a non-negative integer vector of each observed historical data.			
CompStat	a list of four elements that represents the "compatibility(sufficient) statistics" for $\lambda$ . Default is NULL so the fitting will be based on the data. If the CompStat is provided then the inputs in Data.Cur and Data.Hist will be ignored.			
	n0 is the number of observations in the historical data.			
	mean0 is the sample mean of the historical data.			
	n1 is the number of observations in the current data.			
	mean1 is the sample mean of the current data.			
prior	a list of the hyperparameters in the prior for both $\lambda$ and $\delta$ . A Gamma distribution is used as the prior of $\lambda$ , and a Beta distribution is used as the prior of $\delta$ .			
	lambda . shape is the shape (hyper)parameter in the prior distribution $Gamma(shape, scale)$ for $\lambda$ .			
	lambda . scale is the scale (hyper)parameter in the prior distribution $Gamma(shape, scale)$ for $\lambda$ .			
	delta.alpha is the hyperparameter $\alpha$ in the prior distribution $Beta(\alpha,\beta)$ for $\delta$ .			
	delta.beta is the hyperparameter $\beta$ in the prior distribution $Beta(\alpha,\beta)$ for $\delta$ .			
MCMCmethod	sampling method for $\delta$ in MCMC. It can be either 'IND' for independence proposal; or 'RW' for random walk proposal on logit scale.			

rw.logit.delta the stepsize(variance of the normal distribution) for the random walk proposal

of logit  $\delta$ . Only applicable if MCMCmethod = 'RW'.

26 PoissonNPP\_MCMC

ind.delta.alpha

specifies the first parameter  $\alpha$  when independent proposal  $Beta(\alpha,\beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

ind.delta.beta specifies the first parameter  $\beta$  when independent proposal  $Beta(\alpha,\beta)$  for  $\delta$  is

used. Only applicable if MCMCmethod = 'IND'

nsample specifies the number of posterior samples in the output.

control.mcmc a list of three elements used in posterior sampling.

delta.ini is the initial value of  $\delta$  in MCMC sampling.

burnin is the number of burn-ins. The output will only show MCMC samples

after bunrin.

thin is the thinning parameter in MCMC sampling.

#### **Details**

The outputs include posteriors of the model parameter(s) and power parameter, acceptance rate in sampling  $\delta$ , and the deviance information criteria.

#### Value

A list of class "NPP" with four elements:

lambda posterior of the model parameter  $\lambda$ .

delta posterior of the power parameter  $\delta$ .

acceptance the acceptance rate in MCMC sampling for  $\delta$  using Metropolis-Hastings algo-

rithm.

DIC the deviance information criteria for model diagnostics.

# Author(s)

Zifei Han <hanzifei1@gmail.com>

#### References

Ibrahim, J.G., Chen, M.-H., Gwon, Y. and Chen, F. (2015). The Power Prior: Theory and Applications. *Statistics in Medicine* 34:3724-3749.

Duan, Y., Ye, K. and Smith, E.P. (2006). Evaluating Water Quality: Using Power Priors to Incorporate Historical Information. *Environmetrics* 17:95-106.

#### See Also

MultinomialNPP\_MCMC; NormalNPP\_MCMC; BerNPP\_MCMC;

SPDData 27

#### **Examples**

**SPDData** 

Dataset for Diagnostic Test (PartoSure Test, Medical Device) Evaluation for Spontaneous Preterm Delivery

# **Description**

The diagnostic test was developed to aid in rapidly assess the risk of spontaneouspreterm delivery within 7 days from the time of diagnosis in pre-pregnant women with signs and symptoms. The same diagnostic test was used for two populations in US and EU respectively. The number of counts in the four cells (True positive, false positive, false negative, true negative) was recorded.

#### Usage

```
data("SPDData")
```

# Format

A data frame with 2 observations on the following 5 variables.

Data. Region region where the diagnostic test was conducted

TPDP number of subjects with tested positive and the disease status positive (true positive)

TPDN number of subjects with tested positive but the disease status negative (false positive)

TNDP number of subjects with tested negative and the disease status positive (false negative)

TNDN number of subjects with tested negative and the disease status negative (true negative)

#### Source

https://www.accessdata.fda.gov/cdrh docs/pdf16/P160052C.pdf

#### **Examples**

```
data(SPDData)
```

28 VaccineData

VaccineData	Dataset of a Vaccine Trial for RotaTeq and Multiple Historical Trials for Control Group

# Description

The study was designed to investigate the concomitant use of RotaTeq(Test Vaccine) and some routine pediatric vaccines between 2001-2005. The dataset includes four historical control trials. The purpose of the study is to borrow the historical controls for the non-inferiority trial. The interest is in the response rate to the routine vaccines.

#### Usage

```
data("VaccineData")
```

#### **Format**

A data frame with 6 observations on the following 7 variables.

Data. Time indicator of historical data (coded as 0) or current data (coded as 1).

StudyID character to distinguish different studies.

Group indicator of control group (coded as 0) or treatment group (coded as 1).

Start. Year start year of the trial

End. Year end year of the trial

N total number of patients enrolled and dosed in the group

y total number of patients respond to the vaccine

#### References

Liu, G.F. (2018). A Dynamic Power Prior for Borrowing Historical Data in Noninferiority Trials with Binary Endpoint. *Pharmaceutical Statistics* 17:61-73.

# **Examples**

data(VaccineData)

# **Index**

```
*Topic Bernoulli
                                                  PHData, 24
    VaccineData, 28
                                                  PoissonNPP_MCMC, 4, 7, 21, 24, 25
*Topic multinomial
                                                  SPDData, 27
    SPDData, 27
*Topic normalized power prior
                                                  VaccineData, 28
    BerNPP_MCMC, 2
    LaplacelogC, 4
    LMNPP_MCMC, 5
    logCdelta, 7
    logCknot, 8
    loglikBerD0,9
    loglikNormD0, 10
    ModeDeltaBerNPP, 11
    ModeDeltaLMNPP, 12
    ModeDeltaMultinomialNPP, 14
    ModeDeltaNormalNPP, 16
    ModeDeltaPoisNPP, 18
    MultinomialNPP_MCMC, 19
    NormalNPP_MCMC, 22
    PoissonNPP_MCMC, 25
*Topic normal
    PHData, 24
BerNPP_MCMC, 2, 7, 21, 24, 26
LaplacelogC, 4
LMNPP_MCMC, 5
logCdelta, 7, 8–10
logCknot, 5, 8, 8, 9, 10
loglikBerD0, 8, 9, 10
loglikNormD0, 8, 9, 10
ModeDeltaBerNPP, 11, 14, 15, 17, 19
ModeDeltaLMNPP, 12
ModeDeltaMultinomialNPP, 12, 14, 14, 17, 19
ModeDeltaNormalNPP, 12, 14, 15, 16, 19
ModeDeltaPoisNPP, 12, 15, 17, 18
MultinomialNPP_MCMC, 4, 7, 19, 24, 26
NormalNPP_MCMC, 4, 7, 21, 22, 26
```